THE TWENTY-FIRST SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

By invitation of Brown University, the twenty-first summer meeting of the Society was held at that institution on Tuesday and Wednesday, September 8–9, 1914, in connection with the celebration of the one hundred and fiftieth anniversary of the founding of the university. Two sessions were held on Tuesday and a morning session on Wednesday, the following fifty-two members being in attendance:

Professor R. C. Archibald, Dr. Ida Barney, Professor G. D. Birkhoff, Professor J. W. Bradshaw, Professor B. H. Camp, Dr. A. B. Chace, Dr. A. Cohen, Professor F. N. Cole, Dr. C. F. Craig, Mr. C. H. Currier, Mrs. E. B. Davis, Professor E. W. Davis, Professor S. C. Davisson, Professor W. P. Durfee, Professor L. P. Eisenhart, Mr. H. J. Ettlinger, Professor G. C. Evans, Professor H. B. Fine, Professor T. S. Fiske, Dr. W. C. Graustein, Professor C. N. Haskins, Professor E. R. Hedrick, Professor E. V. Huntington, Professor W. A. Hurwitz, Dr. Dunham Jackson, Dr. R. A. Johnson, Professor Edward Kasner, Professor C. J. Keyser, Mr. John McDonnell, Dr. J. K. Lamond, Professor H. P. Manning, Professor G. A. Miller, Professor G. D. Olds, Dr. F. W. Owens, Professor Alexander Pell, Professor Anna J. Pell, Professor A. D. Pitcher, Professor T. M. Putnam, Professor R. G. D. Richardson, Professor E. D. Roe, Jr., Mrs. E. D. Roe, Jr., Professor D. A. Rothrock, Dr. H. M. Sheffer, Professor H. E. Slaught, Professor E. R. Smith, Professor J. N. Van der Vries, Professor E. B. Van Vleck, Professor A. G. Webster, Professor J. H. M. Wedderburn, Professor H. S. White, Professor Ruth G. Wood, Professor J. W. Young.

The President of the Society, Professor E. B. Van Vleck, occupied the chair at the morning sessions; Vice-President L. P. Eisenhart presided at the Tuesday afternoon session. The Council announced the election of the following persons to membership in the Society: Mr. Lincoln K. Adkins, University of Minnesota; Dr. Lennie P. Copeland, Wellesley College; Mr. John W. Cromwell, Jr., Washington, D. C., High Schools; Professor Tsuruichi Hayashi, Tôhoku Imperial University, Sendai, Japan; Professor Claude I. Palmer, Armour

Institute of Technology; Mr. George A. Pfeiffer, Columbia University; Mr. Paul R. Rider, Yale University; Dr. Alfred Rosenblatt, University of Cracow; Miss Caroline E. Seely, Columbia University. Eleven applications for membership in the Society were received.

In order that those wishing to do so may attend the winter meeting of the Chicago Section, the meeting of Section A of the American Association, and the annual meeting of the Society (at which President Van Vleck will deliver his presidential address), it was voted to suspend By-Law III to the extent of holding the annual meeting not later than January 5, 1915. A committee, consisting of Professors M. W. Haskell, H. F. Blichfeldt, L. E. Dickson, O. D. Kellogg, and the Secretary, was appointed to arrange for holding the summer meeting at San Francisco in 1915.

It was decided to issue only a List of Officers and Members next year, instead of the complete Annual Register.

The first session, on Tuesday morning, opened with an address of welcome by Chancellor A. B. Chace of Brown University. The social programme included a tea in the John Carter Brown Library on Tuesday afternoon and luncheon in Rockefeller Hall on Wednesday, both given by Professor N. F. Davis. The University gave a dinner to the Society at the University Club on Tuesday evening, which was concluded by an address by President Faunce and an interesting account by Professor Carl Barus of the "Historical development of the modern theories of physics." A vote of thanks was tendered to the University and its officers for their generous hospitality. Wednesday afternoon was devoted to an excursion to Newport.

The following papers were presented at this meeting:

(1) Dr. F. M. MORGAN: "A plane cubic Cremona transformation and its inverse."

(2) Professor L. P. EISENHART: "Conjugate systems with equal tangential invariants and the transformation of Moutard."

(3) Professor C. E. LOVE: "Singular integral equations of the Volterra type."

(4) Professor O. E. GLENN: "Modular invariant processes."

(5) Professor L. E. DICKSON: "Invariants, seminvariants and covariants of the ternary and quaternary quadratic form modulo 2."

(6) Professor L. E. DICKSON: "The points of inflexion of a plane cubic curve."

(7) Professor L. E. DICKSON: "A fundamental set of modular invariants of the system of the binary cubic, quadratic, and linear form."

(8) Professor L. E. DICKSON: "Invariants in the theory of numbers."

(9) Professor F. B. WILEY: "Proof of the finiteness of the modular covariants of a system of binary forms and cogredient points."

(10) Professor E. V. HUNTINGTON: "The theorem of rotation in elementary dynamics."

(11) Dr. R. D. BEETLE: "Congruences associated with a one-parameter family of curves."

(12) Professor G. C. EVANS: "The non-homogeneous parabolic differential equation."

(13) Dr. R. A. JOHNSON: "The conic as a space element."

(14) Professor W. A. HURWITZ and Dr. L. L. SILVERMAN: "On the consistency and equivalence of certain definitions of summability."

(15) Professor MAXIME BÔCHER: "The method of successive approximations for linear differential systems."

(16) Professor MAXIME BÔCHER: "The smallest characteristic numbers in a certain exceptional case."

(17) Professor B. H. CAMP: "On the series obtained by term-wise integration."

(18) Professor G. A. MILLER: "On the φ-subgroup of a group."

(19) Dr. T. E. MASON: "On functions transcendentally transcendental with respect to a given realm of rationality."

(20) Dr. T. E. MASON: "Mechanical device for testing Mersenne numbers for primes."

(21) Mr. H. S. VANDIVER: "On Bernoulli's numbers, Fermat's quotient and last theorem."

(22) Professor L. C. KARPINSKI: "An early algorism."

(23) Professor H. S. WHITE: "Triple-systems on 31 letters; a reconnoissance."

(24) Miss L. D. CUMMINGS: "The trains for 42 non-congruent triple-systems on 15 elements."

(25) Professor J. H. M. WEDDERBURN: "On matrices whose coefficients are entire functions."

(26) Professor E. R. SMITH: "A problem in the fitting of polynomial curves to certain kinds of data."

(27) Dr. H. R. KINGSTON: "Metric properties of nets of plane curves."

(28) Professor G. D. BIRKHOFF: "The iterated transformation of a plane into itself."

(29) Professor W. B. FITE: "Prime power groups in which every commutator of prime order is invariant."

(30) Professor EDWARD KASNER: "Transversality for double integrals in the calculus of variations and for contact transformations."

(31) Professor EDWARD KASNER: "The decomposition of conformal transformations into factors of period two."

(32) Professor R. G. D. RICHARDSON: "A new boundary value problem for linear hyperbolic differential equations of the second order."

(33) Dr. JOSEPH ROSENBAUM: "Mixed linear integral equations over a two-dimensional region."

(34) Professor D. C. GILLESPIE: "Cauchy's definition of a definite integral."

Dr. Kingston's paper was communicated to the Society by Professor Wilczynski. Dr. Rosenbaum was introduced by Professor Hurwitz. In the absence of the authors, Miss Cummings's paper was read by Professor White and the papers of Dr. Morgan, Professor Love, Professor Glenn, Professor Dickson, Professor Wiley, Dr. Beetle, Professor Bôcher, Dr. Mason, Mr. Vandiver, Professor Karpinski, Dr. Kingston, Professor Fite, and Professor Gillespie were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper Dr. Morgan studies the properties of the plane cubic Cremona transformation $S = (_{y_1(x_1^2+y_1^2)}^{x} \; _{x_1(x_1^2+y_1^2)}^{y} \; _{x_1y_1x_1}^{z})$ and its inverse $T = (_{xy^2}^{x_1} \; _{yz^2}^{y_1} \; _{z(x_1^2+y_1^2)}^{z_1})$. Both of these transformations are examples of an α_i coinciding with an α_k. When written in non-homogeneous coordinates these transformations present a simple ruler and compass construction by means of which one may construct a cissoid, a four-leaved rose, a lemniscate, and other higher plane curves.

2. When the tangential coordinates of a surface S are solutions of an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} + \frac{\partial \log \rho}{\partial v}\frac{\partial\theta}{\partial u} + \frac{\partial \log \rho}{\partial u}\frac{\partial\theta}{\partial v} = 0 \tag{1}$$

the parametric coordinate system is said to have equal tangential invariants. Professor Eisenhart shows that there exist transformations of S into surfaces S_1 such that the parametric system on S_1 is conjugate with equal tangential invariants, and moreover the lines joining corresponding points on S and S_1 form a congruence whose developables meet these surfaces in the parametric curves. In this case the surfaces S and S_1 are said to be in the relation of a transformation Ω. The surfaces associate to S and S_1 with the asymptotic lines of the former corresponding to the parametric conjugate systems on the latter can be so placed in space that they are the focal surfaces of a W-congruence. If S and S_1 are in the relation of a transformation Ω, the tangent planes to these surfaces at corresponding points are harmonic to the focal planes of the congruence of lines of intersection of these planes. Of particular interest is the case in which the latter congruence is normal. To this class belong the transformations of surfaces of Voss discussed by the author in the July number of the *Transactions*. The transformations of surfaces with isothermal spherical representation of their lines of curvature established in the ninth volume of the *Transactions* are transformations Ω and the only ones such that S and S_1 are the two sheets of the envelope of a two-parameter family of spheres. If S_1 and S_2 are two transforms of S, there exist an infinity of surfaces S' each of which is in the relation of a transformation Ω with S_1 and S_2. Moreover, if M, M_1, M_2 are the corresponding points of S, S_1, S_2, the corresponding points of the surfaces S' lie on a conic in the plane of M, M_1, M_2; and the tangent planes to the surfaces S' pass through the point of intersection of the corresponding tangent planes to S, S_1, S_2.

The transformation of Moutard of equations of the type (1) plays an important rôle in the transformations Ω, as also in the transformations K of conjugate systems with equal point invariants announced by the author at the April meeting of the Society. The paper will appear in the *Palermo Rendiconti*.

3. Professor Love's paper proves the existence of a continuous solution $\varphi(x)$ of the singular Volterra equation of the second kind

$$\varphi(x) = f(x) + \int_x^\infty K(x, t)\varphi(t)dt$$

under various hypotheses concerning the functions $f(x)$ and $K(x, t)$. In each case the behavior of $\varphi(x)$ for large values of x is also considered. The paper will appear in the October number of the *Transactions*.

4. Under linear transformations on the variables x_i $(i=1, \cdots, m)$ whose coefficients are positive residues of a prime number p the functions $x_i^{p^t}$ $(i = 1, \cdots, m)$ are cogredient to the variables. Professor Glenn proves this and shows that it leads to the fact that the polar operator

$$x_1^{p^t}\frac{\partial}{\partial x_1} + \cdots + x_m^{p^t}\frac{\partial}{\partial x_m},$$

applied to any covariant of an algebraic m-ary form f, gives a formal modular covariant of f. Modular transvectants are also defined by means of such cogredient expressions, and their properties discussed. Application is made to the derivation of an irreducible system of formal modular concomitants of a linear form, consisting of two invariants and six covariants (cf. Dickson, Madison Colloquium Lectures, page 40).

Since the transformation on the x's induces a similar group of transformations on the coefficients $a_0, a_1, \cdots$, we have a dual theory of the modular Aronhold operators

$$a_0^{p^t}\frac{\partial}{\partial a_0} + \cdots + a_\mu^{p^t}\frac{\partial}{\partial a_\mu}.$$

5. The object of the first paper by Professor Dickson is to provide an improved method for finding the leading coefficient of a modular covariant of a form or forms in three or more variables. For the case of a ternary quadratic form modulo 2, the method employed in the Madison Colloquium Lectures, pages 77–79, is sufficiently brief, but is not as expeditious as the new method, especially in more difficult problems. The paper will appear in the BULLETIN.

6. In the second paper by Professor Dickson, an elementary and largely novel exposition is given of the classic theory of the inflexion points of a plane cubic curve without singular points. Then the Galois group of the equation X for the abscissas of the nine inflexion points is determined for special cubic curves as well as for the curve in whose equation the

coefficients are independent variables. The knowledge of the group and its properties is applied to a complete analysis of the solution by radicals of the equation X. The paper will appear in the December number of the *Annals of Mathematics*.

7. The third paper by Professor Dickson shows how simple and remarkably effective is the theory of modular invariants based upon the separation of equivalent sets of ground forms into classes (cf. Madison Colloquium Lectures), by treating the simultaneous modular invariants of the binary cubic, quadratic, and linear form. The paper will appear in the *Quarterly Journal of Mathematics*.

8. A. Hurwitz* introduced the concept of formal invariants of binary forms (whose coefficients are independent variables) under the group of all binary linear transformations with integral coefficients taken modulo p, a prime. Aside from the cases† of quadratic and cubic binary forms, and a theorem by Miss Sanderson‡ connecting the subject with modular invariants, no progress has been made towards a general theory. A remarkably simple and effective method of constructing formal invariants is given in the fourth paper by Professor Dickson. In the case of a single form f in n variables, the method consists in forming the symmetric functions of certain like powers of the values of f at the real points in modular space. The power used is such that f has the same value at $(x_1, \cdots, x_n)$ as for $(\rho x_1, \cdots, \rho x_n)$, where ρ is any integer. In the case of several forms f_i, we employ the symmetric functions of $\phi(v_1, v_2, \cdots)$, in which $v_1, v_2, \cdots$ range over the sets of values of powers of $f_1, f_2, \cdots$ at the real points, while ϕ is any polynomial with integral coefficients. We may also employ the points whose coordinates are in a Galois field of order p^k. From the special cases examined, it appears that the method furnishes a fundamental system of formal invariants. There is an immediate extension to the case of seminvariants and to the general case of invariants under any modular group. The paper will appear in the *Transactions*.

* *Archiv der Mathematik und Physik* (3), vol. 5 (1903), p. 25.
† Dickson, The Madison Colloquium Lectures, pp. 41–54.
‡ *Transactions*, vol. 14 (1913), p. 490.

9. The question of the finiteness of the modular covariants of a system of forms and cogredient points arises naturally at this time in view of the fact that Professor Dickson has proved (*Transactions*, 1913) the finiteness of the modular covariants of a system of forms without cogredient points and Professor Krathwohl secured in his Chicago dissertation (*American Journal*, October, 1914) a fundamental set of modular covariants of two cogredient binary points. In the present paper Professor Wiley secures the theorem:

The set of all modular covariants of any system of forms in x_{01}, x_{02} and the cogredient points x_{k1}, x_{k2} $(k = 1, \cdots, n)$ is finite in the sense that they are all rational integral functions, with integral coefficients taken modulo p, of a finite number of covariants of the set.

This paper will appear in the *Transactions*.

10. In the theory of the plane motion of a rigid body, the equation $Fr = (W/g)k^2 d\omega/dt$, where $d\omega/dt$ is the angular acceleration of the body at the instant considered, plays an important rôle. In this equation, Fr is the sum of the moments of the external forces and k the radius of gyration of the body about a suitably chosen point Q. The question at once arises, what is meant by a suitably chosen point in this case. All the books agree that the equation is true if Q is fixed in the body and fixed in space, or if Q coincides with the center of mass. Many books state that the equation is true also when Q is the instantaneous center of rotation, and some that it is true when Q is any point whatever. In view of these discrepancies in the current elementary textbooks, it seems worth while to review the facts concerning this important equation. In order to do this most effectively, Professor Huntington states and proves two distinct theorems, one for the case in which Q is fixed in the body, the second for the case in which Q is fixed in the plane.

I. If Q is a point fixed in the body, then the equation $Fr = (W/g)k^2 d\omega/dt$ is true: (1) when Q is fixed in the plane, (2) when Q is moving with constant velocity in a straight line, (3) when Q is the center of mass, (4) when Q has an acceleration whose direction at the instant in question passes through the center of mass; and in no other cases. In particular, it is *not* in general legitimate to "take moments about the instantaneous center."

II. If Q is fixed in space, $Fr = (W/g)d(c\bar{v})/dt + (W/g)k_0^2 d\omega/dt$, where Fr is the sum of the moments of the external forces about the point Q at the instant in question, $c\bar{v}$ is the moment of the velocity of the center of mass about Q at that instant, and k_0 is the radius of gyration about the center of mass.

These two theorems are of course implicitly contained in any standard treatise on the subject; but they do not seem to be explicitly stated in this form in any of the elementary textbooks. The proofs are here given in a very direct and simple form.

11. With a one-parameter family of curves may be associated a number of rectilinear congruences. The congruence formed by the tangents to the curves has been very extensively studied, and there are many theorems concerning the relations between the properties of this congruence and those of the curves and of the surface on which they lie. The other congruences naturally associated with a one-parameter family of curves have received little attention. In this paper, Dr. Beetle considers some of these other congruences, giving particular attention to those formed by the principal normals and binormals of the curves.

In connection with certain properties of the congruences, it is of interest to consider those surfaces on which the asymptotic lines in one system, or both, are geodesic parallels. It is shown that the determination of all such surfaces requires the solution of a rather complicated partial differential equation of the fourth order. Several characteristic properties of these surfaces are found.

12. Professor Evans considers the non-homogeneous partial differential equation $\partial u/\partial t - \partial^2 u/\partial x^2 = f(x, t)$ and the function which is its principal solution, when the proper derivatives exist, with the object of generalizing the conditions on $f(x, t)$ necessary for the existence of these derivatives. According to the nature of the parabolic equation, such conditions should be asymmetrical in character. It is proved that those derivatives exist at the point (x_1, t_1) provided that $f(x, t)$ is finite and integrable in a suitably defined region, and at (x_1, t_1) is continuous and satisfies the condition $|f(x, t) - f(x_1, t)| \leqq N|x - x_1|^\tau$, where N and τ are positive numbers, not zero.

13. Dr. Johnson uses as coordinates of the conic in three-

dimensional space the elements of the discriminant of its tangential equation; in other words, a conic is represented by ten homogeneous coordinates which are elements of a vanishing four-rowed symmetrical determinant. Thus there is a one-to-one correspondence between the conics of space and points of a quartic hypersurface in space of nine dimensions. The equations, in terms of these coordinates, which correspond to various projective relations between conics and other figures, such as points, lines, planes, quadrics, are developed for numerous simple cases. After a brief consideration of the relation of apolarity between quadrics, a detailed discussion is given of the possible linear systems of conics.

14. In the present paper Professor Hurwitz and Dr. Silverman, on the ground of previous work by Silverman, Toeplitz, and Schur, establish a correspondence between certain functions $f(z)$ and certain definitions of summability of a divergent sequence. If $f(z)$ is analytic within and on the boundary of the circle C of radius $\frac{1}{2}$ about the point $\frac{1}{2}$, the corresponding definition is a generalization of convergence, i. e., correctly evaluates any convergent sequence. All such definitions are consistent, i. e., if two of them evaluate a certain sequence, they give it the same value. Two definitions are equivalent, i. e., evaluate exactly the same sequences, provided the corresponding functions have the same zeros with the same multiplicities in C. Further properties of the definitions in question are derived; the equivalence of the Cesàro and Hölder definitions of the same order appears as an incidental result. The question of equivalence of different definitions has previously received some slight attention; it is believed that the question of consistency is here studied for the first time.

15. The results partially announced by Professor Bôcher at the Cambridge International Congress in 1912 concerning the scope of the method of successive approximations when applied in a very general manner to ordinary linear differential systems are here developed in detail.

16. Professor Bôcher's second paper appeared in full in the October BULLETIN.

17. It has long been known that, if a series is uniformly convergent, the series obtained by integrating it term-wise

is also uniformly convergent. In recent years more general sufficient conditions for term-wise integrability have been discovered, and finally, by Vitali and W. H. Young, conditions which are both necessary and sufficient. If a series which is not uniformly convergent be integrable term-wise over a variable interval, the new series may not be uniformly convergent, but it has certain other characteristics which may be classified according to which of these sufficient conditions the original series satisfies. Furthermore, successive integrations over the same interval lead to progressively simpler series.

In Professor Camp's paper several general theorems dealing with this subject are developed. A particular result, capable of being stated concisely, is that, if a series may be integrated term-wise in the variable interval (a, x), then the new series may be integrated term-wise, even after being multiplied through by any function of x which is absolutely integrable in Lebesgue's sense.

18. All the operators of any group G can be divided into two categories having no common operator by putting into one category (A) all those operators of G which can occur in one of the possible sets of independent generators and into the other category (B) those which do not have this property. The category (B) is a subgroup of G, and this subgroup was called by G. Frattini the φ-subgroup of G. It is always the direct product of its Sylow subgroups and hence it is always solvable. The φ-subgroup of G may also be defined as the cross-cut of all the possible maximal subgroups of G. A simple example of a φ-subgroup is the subgroup of order p^{m-1} in the cyclic group of order p^m, p being any prime number.

Professor Miller developed a number of theorems relating to the φ-subgroups. In particular, he proved that the order of the φ-subgroup of any abelian group of order p^m is p^{m-s}, whenever p^s is the order of the subgroup generated by all of the operators of order p. A necessary and sufficient condition that the φ-subgroup of an abelian group be identity is that all the Sylow subgroups of this abelian group are of type $(1, 1, 1, \cdots)$. If a group can be represented as a primitive substitution group its φ-subgroup is identity; in particular, the φ-subgroup of every simple group of composite order is identity.

19. Dr. Mason finds conditions that first and second order linear difference and q-difference equations and Riccati difference and q-difference equations have only solutions that are transcendentally transcendental with respect to a general realm of rationality which includes the coefficients. These conditions are expressed in the possibility of certain auxiliary equations having solutions algebraic in the realm of rationality considered. The results for these equations in the realm of rationality x are to be found in papers by Tietze, Stridsberg, and a previous paper by Dr. Mason.

20. Lucas has given a method of testing Mersenne numbers for primes, in which he uses multiplication in the binary system of numeration. Dr. Mason shows how multiplication in this system can be reduced to counting correspondences, and suggests a mechanical device to do the counting and recording.

21. Mr. Vandiver obtains some general theorems respecting Fermat's quotient and Bernoulli's numbers and in particular the following relation:

$$\frac{(m^a - 1)h_a}{a} \equiv \sum_{n=1}^{p-1} y_n n^{a-1} \pmod{p},$$

where $1 < a < p - 1$, $1 < m < p$ (p a prime),

$$y_n \equiv -\frac{n}{p} \pmod{m}, \quad m > y_n \geqq 0,$$

$$h_{2a} = (-1)^{a-1} B_a, \quad h_{2a+1} = 0,$$

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad \text{etc.}$$

By means of this theorem it is then shown that Kummer's criteria in reference to Fermat's last theorem may be replaced by the following conditions:

$$\sum_{n=1}^{p-1} y_n n^{p-2} t^n \equiv 0 \pmod{p} \quad (m = 2, 3, \cdots, p-1).$$

(See notation used in *Crelle*, volume 144, page 314.) A number of other transformations of the Kummer congruences are also given.

22. The Latin treatises on the Hindu art of reckoning which are earlier than the fourteenth century are sufficiently rare so that the addition even of anonymous works to the available literature is desirable. An algorism which bears internal evidence that it was one of the very early treatises on this subject has been found by Professor Karpinski in the British Museum. Royal MS .15 B IX contains the algorism in question; this manuscript is dated by the experts of the museum as of the end of the twelfth or beginning of the thirteenth century. A somewhat similar work, directly related to this algorism, is found in the Egerton MS 2261, fol. 225^b–227^b, and a second copy in Codex Paris, 10252, fol. 68^a–70^b. The copy in Paris, written 1476, was found by Björnbo and is included in his list of medieval mathematical works which is preserved in Stockholm. A study is made of all three of the manuscripts.

This paper will appear in the *Bibliotheca Mathematica.*

23. The number of different triad systems on n letters is known only for the small values of n. Reiss proved that at least one system exists for every value of $n = 6t + 1$ or $n = 6t + 3$. E. H. Moore proved that at least two incongruent systems exist for every such n above 13. Zulauf showed two distinct systems for 13. Miss Cummings has distinguished 24 systems for $n = 15$ without exhausting the possibilities. In the present paper, from one simple theorem and certain data not yet published, Professor White demonstrates the existence of a very large number, above 10^{13}, of systems of a particular kind on 31 letters.

24. Miss Cummings has applied to triple-systems on 15 elements the method for determining the non-congruency of triple-systems given by Professor H. S. White in the *Transactions* of January, 1913. The paper shows the 203 distinct types of trains which occur in 42 non-congruent systems Δ_{15}.

25. In this paper, Professor Wedderburn extends the methods ordinarily used in reducing to its normal form a matrix whose coefficients are polynomials in a single variable so as to make these methods applicable when polynomials are replaced by entire functions or, more generally, by functions holomorphic in a simply connected region.

26. If to a series of values of x which are in an arithmetical progression there corresponds a series of observed values of y, and if it is desired to determine by the method of least squares or by the method of moments the coefficients of the polynomial

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_k x^k$$

which give the best relation between x and y, the problem can be simplified by evaluating the determinant

$$D_k(n) = \begin{vmatrix} \sum_0^n x^0 & \sum_0^n x & \sum_0^n x^2 & \ldots & \sum_0^n x^k \\ \sum_0^n x & \sum_0^n x^2 & \sum_0^n x^3 & \ldots & \sum_0^n x^{k+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \sum_0^n x^k & \sum_0^n x^{k+1} & \sum_0^n x^{k+2} & \ldots & \sum_0^n x^{2k} \end{vmatrix}.$$

By separating the determinant into two factors Professor Smith finds the roots of $D_k(n)$ and consequently its factors. It follows then that when n is less than $k+1$, $D_k(n)$ vanishes, when $n = k+1$, $D_k(n)$ is the square of a simple Vandermondian, and finally when n is greater than $k+1$,

$$D_k(n) = C_k n^k \prod_{v=0}^{v=k-1} (n-k+v)^{v+1}(n+k-v)^{v+1},$$

where C_k is a constant for any value of k. Having found this value for the determinant, simple expressions are obtained for the coefficients $a_0, a_1, a_2, \ldots, a_k$ which depend only on the observed values of y and on the number of the observations.

27. In a memoir entitled "One-parameter families and nets of plane curves,"* Wilczynski has discussed the projective differential properties of nets of plane curves, by means of a completely integrable system of three partial differential equations of the second order. In Dr. Kingston's paper the foundation is laid for a study of the metric differential properties of such nets. In order to accomplish this, it becomes necessary to consider, besides the coefficients of the partial differential equations used by Wilczynski, the coefficients E, F, and G of the square of the element of arc when this is ex-

* *Transactions*, vol. 12 (1911), pp. 473–510.

pressed in terms of the parameters of the net curves as coordinates. All of these quantities taken together, which must moreover satisfy certain relations, determine a net uniquely except for a motion combined with a reflection. The application of the general theory to the particular cases of orthogonal and isothermal nets gives rise to interesting results.

28. The iterated transformation of a plane into itself leads to hitherto untreated problems in point set theory which stand in vital relation to certain dynamical questions. The problem considered by Professor Birkhoff is that of characterizing as far as possible the sequence of positions taken by an arbitrary point of the plane under repetition of the transformation and its inverse. Some of the results obtained run parallel to those given by him in an earlier paper.*

29. Professor Fite's paper concerns itself with those prime power groups in which every commutator of prime order is invariant. It is shown that if G is of order p^m (p an odd prime) and any cogredient has a cyclic central, the second central must contain a commutator of order p that is not invariant in G. In fact, if any cogredient contains a non-invariant operation of order p, the second central must contain a commutator of order p that is not invariant in G. Of various other results obtained the two following may be mentioned: (1) If every operation of the second central whose order does not exceed p^r is invariant in G, every operation of G whose order does not exceed p^r is invariant. (2) If the commutator subgroup is cyclic and the $(k-1)$th central is of class k_1, G must be of class k, where $k \leqq 3k_1$.

30. In connection with the minimizing of double integrals, Kneser's theory of transversality gives rise to a correspondence between the surface elements and the lineal elements at any point of space. This correspondence is not entirely arbitrary in character. Professor Kasner obtains a simple geometric property which furnishes a criterion as to when a given correspondence can be regarded as a transversality correspondence. In particular the only linear correspondences satisfying the test are polarities.

* *Bulletin Soc. Math. de France*, vol. 40 (1912), pp. 305–323.

The general result obtained is also applied to the theory of infinitesimal contact transformations, and indirectly to the transversality correspondences associated with simple integrals in space. Certain connections between simple and double integrals are thus derived.

31. The conformal transformations discussed in Professor Kasner's second paper are assumed to be regular in the neighborhood of the origin. They are thus expressed by power series in $x + iy$ or $x - iy$ without absolute term. It is shown that there are two types of transformations of period two: the first, in $x + iy$, are called conformal involutions; and the second, in $x - iy$, are identified with symmetries (in the general sense of Schwarz). The main results of the paper are criteria as to when a given conformal transformation can be regarded as the product of either two involutions, or two symmetries. The classes of transformations thus obtained are very large, only a finite number of restrictions being imposed on the coefficients of the corresponding power series. As in the author's earlier results in conformal invariants, distinctions arise according to the rationality or irrationality of certain angles determined by the coefficients.

32. For linear hyperbolic differential equations of the second order in two variables, the boundary value problem ordinarily considered is that of finding a solution $u(x, y)$ which on a piece of a curve has prescribed values for u and $\partial u/\partial n$. The simplest case of the new problem studied consists in determining whether there exist parameter values λ of the equation

$$(1) \qquad (pu_x)_x - (qu_y)_y + \lambda u = 0$$
$$[p(x, y) \geqq k_1 > 0, \quad q(x, y) \geqq k_2 > 0]$$

such that the corresponding solutions vanish on the boundary of a region $a_1 \leqq x \leqq a_2$, $b_1 \leqq y \leqq b_2$. Closely associated with this problem are new problems in linear equations and in quadratic forms each with an infinite number of variables, as well as with n variables where n is allowed to increase indefinitely. Professor Richardson has shown that there is a denumerable infinity of parameter values λ of the equation (1) for which there are solutions of the type sought. This set of values has limiting points at both positive and negative

infinity as well as at one or more finite points. Under restrictions similar to those usually imposed, a function $f(x, y)$ may be developed in a series of solutions with Fourier coefficients.

33. Following out methods previously used by Professor W. A. Hurwitz in discussing mixed linear integral equations in one dimension, Dr. Rosenbaum in this paper establishes similar results for the case of two dimensions. The unknown function appears under integral signs operating over a plane region and over curves, and the values of the unknown function at special points also appear in the equation. The adjoint system of equations now involves as unknowns one function of two variables, several functions of one variable, and several constants. The notions of resolvent system, orthogonalization of principal solutions, and pseudo-resolvent system, receive similar generalizations.

34. The Riemann definition of the definite integral of a bounded function involves the values of the function at arbitrary points of the sub-intervals of a scheme of subdivision, while the Cauchy definition involves the values only at ends of sub-intervals. It is obvious that if the Riemann integral exists the Cauchy integral will exist; it is not immediately evident whether the converse is true or false. Professor Gillespie proves in this note that the two definitions are equivalent.

F. N. COLE,
Secretary.

INFINITE REGIONS IN GEOMETRY.

BY PROFESSOR EDWIN BIDWELL WILSON.

(Read before the American Mathematical Society, February 28, 1914.)

THE recent contribution by Professor Bôcher on "The infinite regions of various geometries"* puts me in mind of some ideas which I have long held on this subject and which I desire to offer to readers of his article.

The points which he most desires to make are:

1°, that when we are dealing with other geometries than

* This BULLETIN, vol. 20, pp. 185–200, January, 1914.

the projective we should replace the infinite line of plane projective geometry and the infinite plane of projective geometry in three dimensions by such other infinite region as may be most appropriate to the geometry we are considering, and

2°, that in particular when we introduce transformations which throw points to infinity we have not put our work in satisfactory form until we have made clear what infinite region we assume.

With the second of these points I am entirely in accord, and with the first, also, if it is properly understood; but I am inclined to think that Professor Bôcher has not put the matter of infinity in geometry in the best way* and that a literal adherence to his position would be unfortunate.

Just as he protests against the use of infinity in a carefree manner in different geometries, I wish on my part to protest against its careless use in any particular geometry. According to the view which I believe to be in fullest consonance with modern scientific attitudes toward geometry, there are no infinite (or ideal) points in the projective plane, no line at infinity in projective geometry, no infinite point in real inversive geometry, nor any pair of lines at infinity in complex inversive geometry.

To substantiate this point of view and to show at the same time the relation of my view to Professor Bôcher's I must give some definitions, in particular some definition of geometry. And as my predecessor gave no definition of what he meant by a geometry (except by implication), I shall begin by formulating a definition which seems to me to express his point of view.

Definition 1.—A geometry is the ensemble of those properties of configurations in the euclidean plane (and in its ideal extensions) which are invariant under the transformations of an r-parametered group†

* There can be no doubt that Professor Bôcher is familiar with the views which I shall expound and that he had reasons which seemed best to him for sticking close to the exposition he chose. There is still a different presentation, more elementary than either of ours, which I have heard Professor F. S. Woods offer at a semi-public gathering and which I hope he may offer to the BULLETIN; for in a subject like infinity in geometry I believe that the maximum satisfaction comes only with a multiplicity of views.

† It is only for ease that we restrict the definition to the plane and omit from the statement such groups as that of the Bewegungen und Umlegungen (motions and orthogonal reflections).

$$X' = X'(X, Y; c_1, c_2, \cdots, c_r), \quad Y' = Y'(X, Y; c_1, c_2, \cdots, c_r).$$

Definition 2.—For any particular choice of the parameters such points (X, Y) as render X' or Y' or both infinite are said to be thrown to infinity, and the infinite region of the geometry is of the type of the locus of such points (X, Y) for general values of the parameters.

Except for the unusual precision relative to infinity and a phraseology introduced by Lie, this definition is old. Applied to the characterization of projective geometry it represents the point of view early in the science of projective geometry—Poncelet, Steiner, Chasles, Salmon, Fiedler, and Cayley might reasonably be associated with it. It is still of vital pedagogic importance when we desire to lead the student for the first time from his known euclidean geometry to the new projective. For the recapitulation theory seems here to apply; the natural entogeny is an accelerated phylogeny.

Beginning, however, with von Staudt and continuing through a long succession which we may at present terminate with Veblen and Young,* there has been a tendency to place projective geometry on its own feet, to define it in terms not extraneous to itself. Although such a development has not yet reached many other geometries we are, I think, sufficiently advanced in our point of view to regard the proper definition of a geometry as something like this:†

Definition 3.—Given a system containing a set of undefined symbols (one or more classes of elements, one or more relations) and a set of primitive propositions connecting them; the geometry of this system is constituted of the body of propositions logically deduced from the primitive propositions.

* "A set of assumptions for projective geometry," *Amer. Journ. of Mathematics*, vol. 30, pp. 347–380 (1908).

† See E. V. Huntington's definition of abstract geometry on page 526 of his article "A set of postulates for abstract geometry, expressed in terms of the simple relation of inclusion," *Math. Annalen*, vol. 73, pp. 522–559 (1913). He restricts his definition to cover only the particular geometry he is expounding, but states that such a definition is applicable in other geometries also. It is entirely possible that the definition which we formulate should not be applied in general to a system as yet unformed, but should be applied anew in each particular instance after the system has been formed and then only if the person who forms the system desires to call it a geometry; for in no other way does it seem possible to include all systems which have been or may be called geometries without also including pretty much every deductive system. For restrictions see definitions by B. Russell, Principles of Mathematics, p. 372, and A. N. Whitehead, The Axioms of Projective Geometry, p. 5.

(If we desire to define a geometry as distinguished from a class of geometries, we should insist that the system be categorical.)

Definition 4.—Any class of elements selected from that class, or from one of those classes,* which enters into the definition of a geometry may be called a region of the geometry. The infinite region of the geometry would be, if it existed, a certain special region. The specialization would have to be effected by means of some property which belonged to the geometry and to which the concepts finite and infinite (in some of their many senses) were alternatively applicable.†

Why, upon the basis of Definitions 3 and 4, has the projective plane or projective geometry no infinite region? Simply because if we take any purely projective definition of projective geometry, there are no special regions whatsoever which are singled out from the rest of the plane; a fortiori there can be no infinite region. One of the things which we should be most careful to impress upon the student of projective geometry is that the projective plane is entirely homogeneous.

For a similar reason real euclidean geometry has no infinite region; it has no special points or lines in it. We may refer to Huntington's paper previously cited. The non-euclidean geometry developed in detail by Lewis and myself for the representation of the principle of relativity is also without an infinite region.‡

As to inversive geometry it may well be that as yet there is no exact formulation in postulates, but the possibility of such formulation is so evident that it is safe to say that inversive geometry has no infinite element or infinite region; the inversive plane is homogeneous. We might as well maintain that the surface of a sphere or spherical geometry, when defined by a system of postulates appropriate to the geometry,§

* We could define mixed regions by collecting elements from different classes.

† We formulate no precise definition of infinite elements or infinite region because in most of the geometries which have been handled in the modern logical manner there is no infinite region.

‡ Wilson and Lewis, "The space-time manifold of relativity," *Proceedings of the American Academy*, vol. 48, pp. 389–507. There are singular loci in the geometry, but no special points. If we consider as fundamental the right line and angle, we could regard the singular lines as infinite elements; but the advantage of such procedure is problematical.

§ For a definition which is not proved to be either complete or categorical see E. B. Wilson, "Seven lectures on spherical geometry," *Amer. Math. Monthly*, serially in 1904. In those lectures I made use of the idea of a

possessed a special element (such as a north pole) as that the inversive plane had a special element such as the infinite point.*

As now, speaking from the geometric and logical viewpoints, we have abolished the infinite regions in some, and could abolish them in all, of the geometries of which Professor Bôcher speaks, it is necessary to answer the question: What and where are the infinite regions of which he speaks? They are ideal, they are perhaps nowhere. They arise algebraically through the becoming infinite of some function; this is accidental to our choice of coordinates.† They originate geometrically from the breaking down of a correspondence between the planes of two different geometries;‡ this latter is the really essential geometric fact.

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group of transformations in formulating the axioms, just as Lewis and I later adopted that point of view in our geometry of relativity, loc. cit. In view of the importance of the group concept in modern geometry it might be desirable that some of our eminent specialists in postulates should construct systems in which that concept was emphasized. It is unfortunate in some respects that groups have been tied so closely to analytical representations as far as their geometric uses go.

* Professor Bôcher in a footnote calls attention to the fact that in Study's long treatment of Das Apollonische Problem, *Math. Annalen*, vol. 49, pp. 497–542, there is not a word said concerning the nature of the infinite region. According to our reading of Study's work the reason that no mention of the infinite region occurs may well be that for him, as for us, there is no such region in the geometry of inversion. We may be reading our own ideas into Study's text (which would be a heinous offence on our part toward so illustrious a geometer), but we believe that he makes his point of view quite clear. He does not set up a categorical system of postulates for inversive geometry, but he does point out very precisely that one of the chief differences between his geometry and Mascheroni's is that the latter uses the center of a circle whereas he makes no use of it. As one of Study's fundamental constructions is to find the inverse of any point with respect to any circle, the center could be found if the point at infinity were in the system, and then there would be but little gain in banishing the use of the center. Furthermore Study makes very strong the point of view which actuates him in this article, as in other of his work, namely, the desire to remain completely within his geometry.

† It is interesting to observe that we cannot represent the points on a simple closed curve by a continuously varying parameter so that the correspondence shall be one-to-one. What we do is to use a parameter t subject to $t_0 \leqq t \leqq t_1$, which assigns to the same point the values t_0 and t_1 (to drop t_0 or t_1 from the interval would be to render the correspondence discontinuous). Or we write $-\infty < t < \infty$ and identify ∞ and $-\infty$, and thus introduce the singularity ∞. This is no singularity of the curve. It is merely an unhappy lack of correspondence between arithmetic and geometric types of order. H. B. Phillips and C. L. E. Moore in their "Algebra of plane projective geometry," *Proceedings of the American Academy*, vol. 47, pp. 737–790, probably would have been glad to avoid infinity; but with a non-homogeneous algebra that was out of the question.

‡ One of these being euclidean in the cases Professor Bôcher discusses.

If we start with non-homogeneous (rectangular cartesian) coordinates and write down the equations of a general linear transformation, we get introduced to infinity through the vanishing of the denominator which is common to the two fractions. However, this means merely that from the point of view of projective geometry and the projective plane, we have made a very injudicious start—howsoever judicious the start may have been for a modulation of a pedagogical nature from metric to projective geometry for the student first approaching the latter. Had we selected a triangle with a unit point, introduced trilinear coördinates based on a projective number system, and used the homogeneous form of the transformation, such as is almost invariably used in invariant theory, we should never have met any infinity.

We believe in the pedagogic modulation, but we believe also that particular stress should be laid on the fact that the ideal elements which are introduced are ideal elements of the extended euclidean plane and of extended euclidean geometry rather than of projective geometry, and that they are introduced in or, better, adjoined to the euclidean plane for the purpose of bringing about a correspondence (one out of infinitely many) between the euclidean and projective planes, not only for pedagogic purposes, but rather especially for the sake of carrying across theorems from either geometry into the other. The process of throwing some line, a perfectly normal line of the projective plane (we cannot say a finite line because there is no distinction of finite and infinite), into the ideal region of the euclidean plane is of great use in saving a new demonstration of certain theorems.

When we turn to circle geometry a similar state of affairs is found. We are able to set up a correspondence between the euclidean and circular planes which is one-to-one, points corresponding to points and circles to circles (with proper qualifications), except that there is an extra point in the circular plane for real geometry and two extra imaginary lines intersecting in a real point in the case of the complex circular geometry. We promptly adjoin these as ideal elements in the euclidean plane for the sake of perfecting the correspondence—the reasons being as before partly pedagogic, partly lexicographic.*

* It would indeed be an interesting study in euclidean geometry to take the general solution of the Apollonian problem as developed in inversive

It is somewhat doubtful whether we should regard the ideal regions thus adjoined to the euclidean plane as lying in that plane; it may be better to regard them as lying nowhere at all. Two reasons for this occur to anybody at once. First, the postulates upon which euclidean geometry has been built up are in many cases no longer true for the extended euclidean plane. The introduction of the ideal elements into the euclidean plane has simplified certain statements, namely, such as are essentially projective or inversive (as the case may be), but it has greatly complicated others, which are essentially euclidean. Second, the two cases we have considered show that different ideal regions have to be adjoined in different cases, and that these different ideal regions are mutually incompatible so that they cannot coexist.

The problem of correspondence between the projective and the euclidean planes or between the inversive and euclidean planes is simple by virtue of the fact that the planes do not really clash, in each case we have merely to remedy a defect in the euclidean plane. A vital reason, too, for the naturalness of the correspondence lies in the fact that the euclidean group

$$p, \quad q, \quad yp - xq$$

is a subgroup of the projective group

$$p, \quad q, \quad yp - xq, \quad yp + xq, \quad xp + yq, \quad xp - yq,$$
$$x^2p + xyq, \quad xyp + y^2q,$$

and of the inversive group

$$p, \quad q, \quad yp - xq, \quad xp + yq,$$
$$2xyp + (y^2 - x^2)q, \quad (x^2 - y^2)p + 2xyq.$$

When we try to establish a correspondence between two geometries which are not related in such a manner that one is a subgroup of the other, the matter is not so simple. How could we set about mapping the projective plane on the inversive plane or vice versa? It is futile to content ourselves with trying to map the points of one upon the points of the other; we must be able to carry over certain configurations or

geometry and see how many special cases we might get by selecting different points of the figure for relegation to infinity, and the Apollonian problem for euclidean geometry is not solved until all possibilities have been enumerated.

relations. But projective geometry deals with lines, conics, and so on, whereas inversive geometry deals with circles, cyclides, and so forth. As a problem in abstract geometry, in logic, it appears somewhat difficult to set up a satisfactory and useful relationship between the planes.*

We may overcome the difficulty very readily by coming down to the common subgroup, to euclidean geometry,† and make the transfer between projective and inversive geometry by the intermediary of euclidean. We notice now, however, that the projective and inversive planes clash in their requirements for extensions of the euclidean plane and we shall be on more comfortable ground if we keep the intermediary instead of attempting to cast it aside and obtain a direct correspondence. Projective geometry can be forced into the inversive mould, or inversive geometry into the projective mould only by cracking the mould or the geometry; but if we insist on making the correspondence direct, it would be difficult to say why we should locate the crack at infinity in either geometry.

The definitions of geometry and of infinite region which we have attributed to Professor Bôcher (Definitions 1 and 2) suggest at once the methods of Sophus Lie, and it is an interesting question to ask whether in plane geometry we are bound to introduce for different groups other regions at infinity than those which arise in the projective and inversive groups. Now Lie has tabulated the varieties of groups which occur in plane geometry in twenty-six entries.‡ We may integrate and determine the finite equations of the group. I have done this in a great many cases and have not found any other types of region.§ In space‖ we have a considerable variety—a plane,

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* Abstractly a similar difficulty exists in the case of the projective (or inversive) plane and the euclidean plane; for the projective and euclidean lines are different, the inversive and euclidean circles are not the same—but the differences are not so serious, there are marked similarities as well as differences.

† Indeed to what Klein calls the Hauptgruppe, p, q, $yp - xq$, $xp + yq$, which contains similitude transformations; but euclidean geometry is more familiar.

‡ Continuierliche Gruppen, p. 360.

§ This does not mean that in all similar groups, that is, in all reducible to a common type by a proper choice of variables, the infinite region would be the same; it merely means that a choice of variables exists for which the infinite region becomes as indicated.

‖ Transformationsgruppen, vol. 3, pp. 122–178.

two planes intersecting in a line, three planes with a point in common, a quadric cone,* and perhaps other cases.

We have only one or two more illustrations to give toward substantiating our point of view that the introduction of infinity as Professor Bôcher does is not entirely satisfactory from the geometric point of view, and indeed violates established nomenclature.

Suppose that we consider the group

$$yp - xq, \quad x^2p + xyq - p, \quad xyp + y^2q - q.$$

This is a projective subgroup which leaves the circle $x^2 + y^2 - 1 = 0$ invariant, and is closely associated with Lobachevskian geometry.† According to ordinary nomenclature the fixed circle is the absolute and the absolute is at infinity. Not so, however, if we must determine the finite equations of the group and see what nature of locus is relegated to the bourne beyond the euclidean plane.‡

Suppose that we consider the group

$$yp - xq, \quad x^2p + xyq + p, \quad xyp + y^2q + q.$$

This is a projective subgroup§ which leaves $x^2 + y^2 + 1 = 0$ invariant, and is closely associated with Riemannian geometry. According to ordinary nomenclature lines in this geometry are closed and have a finite length (at least in the real plane) and there does not arise the question of infinity. Not so, however, if again we must determine the finite equations of the group and observe what manner of locus it is that yields infinite values for the transformed coördinates.

The case of real Lobachevskian geometry is illuminating. When that geometry is defined by a set of postulates (see Coolidge, Non-Euclidean Geometry) there is no infinite

* If we may call the minimum cone in inversive space geometry a quadric cone. Can we appropriately, except in euclidean geometry, call it a minimum cone?

† See Klein-Fricke, Automorphe Funktionen. We can also, as Klein has pointed out, use a subgroup of inversive geometry; this is conformal but lines become circles orthogonal to a fundamental circle.

‡ This line lies in what is called the transfinite or ultra-infinite (as contrasted with finite or infinite) region. See Coolidge, Non-Euclidean Geometry, p. 85.

§ With an appropriate change of variable, the group may be made a subgroup of the inversive group, as suggested in the second footnote above. Indeed the groups which leave $x^2 + y^2 \pm 1 = 0$ invariant are

$$yp - xq, \quad (x^2 - y^2)p + 2xyq + p, \quad 2xyp + (y^2 - x^2)q \pm q.$$

region at all, and to modify the postulates so as to introduce any points at infinity would be an unpleasant complication resulting in no gain to the geometry. When, however, we desire to map the projective (or inversive) and Lobachevskian planes one upon the other, we find that the Lobachevskian plane lies entirely within a conic of the projective plane (and entirely upon one side of a circle in the inversive plane—there is here no distinction between inside and outside). To perfect the correspondence we adjoin to the Lobachevskian plane the conic (or circle) as an infinite region and the region outside the conic (or upon the other side of the circle) as an ultra-infinite region.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
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FAMOUS PROBLEMS OF GEOMETRY.

"*Squaring the Circle*," *A History of the Problem.* By E. W. HOBSON. Cambridge, at the University Press, 1913. iii+57 pp. Price 3s.

A FASCINATING and voluminous volume could be written on ancient problems of geometry, their influence on the progress of mathematics and the various developments in mathematics which contributed to their generalization or final settlement.

There is the familiar problem,* to draw from a given point P a line such that the line segment cut off by two intersecting lines l_1, l_2 shall be of given length. This problem is capable of solution with ruler and compasses in but one case, namely when P is on a bisector of an angle between l_1 and l_2. Suppose this condition to obtain. The problem is not an easy one, in general, but Apollonius (about 225 B.C.), known to his contemporaries as the "great geometer," found an elegant solution.† The complete discussion for the case of l_1 and l_2 at right

* Cf. my paper "Discussion and history of certain geometrical problems of Heraclitus and Apollonius," *Proc. Edinb. Math. Soc.*, vol. 28 (1909–1910), pp. 152–178.

† Although twice proposed in the *American Math. Monthly* (Feb., 1910, vol. 17, p. 48, cf. pp. 140–141; Feb., 1911, vol. 18, p. 44, cf. pp. 114–115) no solution has been forthcoming.

angles, was known to Heraclitus. When P was not on a bisector, the Greeks solved the problem by means of: (*a*) the conchoid of Nicomedes, and (*b*) an hyperbola. More than a thousand years elapsed; then we find Girard (in 1629) dealing with the Heraclitus case and led to an equation of the fourth degree. He gives geometrical interpretation of negative roots, a notable achievement in his time. A few years later (1637) Desçartes considered the same problem more generally. Huygens found many solutions, those which are geometrical of the Apollonian case being of remarkable elegance. Newton with keen penetration showed how, with appropriate choice of unknown, the problem leads to the solution of a quadratic equation, whereas, without suitable choice of unknown, we get a reducible biquadratic. Any complete discussion of the problem must also refer to L'Hospital, Ghetaldi, Gergonne, d'Omerique, Steiner, and to the solution of the general problem through methods of projective geometry by G. Russo.

Then recall the "problem of Apollonius," to describe a circle tangent to three coplanar circles, with its many special cases, how it led to developments in geometry.* In 1600 Viète treated the problem by synthetic methods; analytical solutions were found by Newton, Euler and N. Fuss. Fermat discussed, synthetically, the analogous problems for spheres. In later day a solution of this problem was found by pupils of Monge, who discussed the motion of a variable sphere tangent to three given spheres. Dupuis discovered and Hachette proved (1804) that the middle point of the sphere moves on a conic, and that its points of contact describe circles. Soon after (1813) Dupin published his researches on the remarkable surfaces which envelope such a variable sphere, surfaces which he afterward named cyclides. Then followed the ideas of radical axes, power of circles and spheres, similitude, and inversion, connected with developments of Gaultier, Poncelet, Quetelet, Steiner, Plücker, Magnus, Ingram, and Stubbs.

Again, there is the less known problem of Pappus, to inscribe in a circle a triangle whose sides (or sides produced) shall pass through three given points on a line. This is solved by Pappus as a lemma useful for attacking the problem of Apollonius (in just what way is an interesting speculation).†

* Cf. Reye, Synthetische Geometrie der Kugel und linearer Kugelsysteme. Leipzig, 1879.

† Cf. my paper entitled "Centers of similitude of circles and certain theorems attributed to Monge. Were they known to the Greeks?", which is soon to appear in the *Amer. Math. Monthly*.

The generalization of the problem to the case where the given points were any points was found by Simson (1731). After thirty years of attention Castillon gave a solution to the Berlin Academy in 1776. Lagrange, who was present at the meeting, handed a trigonometric solution to Castillon on the next day "apparently for the purpose of showing that an analyst could solve in a single evening a problem that had confessedly required from a geometer of no mean powers many years of study." Then Lexell, Euler, and Fuss discussed the problem but no method was found which could be generalized so that a polygon could be substituted for a triangle. This method was discovered by Giordano di Ottajano, a student at Naples in 1784, when only 16 years of age. Malfatti, Romanus, Lhulier, Carnot, and Meyer Hirsch made later contributions to the discussion. William Wallace, famous for his discovery of Wallace's line (incorrectly called Simson's line), first pointed out (in 1798) indeterminate cases in the triangle problem. To follow up developments in the English school: Lowry made obvious generalization of the problem (in 1806) to a polygon inscribed in an ellipse; Hearn considered the general dual problem of circumscription (instead of inscription) due to Gergonne; and then there are researches of Townsend, Potts, and Renshaw leading up to results of Sir William Rowan Hamilton regarding polygons inscribed in a sphere, an ellipsoid and an hyperboloid, and with sides passing through given points.* But in the early part of the nineteenth century in the French school we find that Gergonne, Servois, Poncelet, Brianchon, and others discussed various phases of the problem of polygons in conics, some of them involving the newly discovered principle of duality.

Similar summary sketches of many other notable problems, with like origin, might be given; among these the most famous are the three problems of the ancients, known under the titles: the duplication of the cube, the trisection of an angle,† and the

* Hamilton considered the solution of these problems which depended on a linear equation in finite differences as an especially tough piece of work. Cf. Life of Sir Wm. R. Hamilton, by R. P. Graves, vol. 3 (1889), pp. 88, 426.

† Who first proved the impossibility of the classic problem of trisection of an angle? I have not met with a statement of this fact in any of the mathematical histories, but surely it was before 1852, when Sir William Rowan Hamilton wrote to De Morgan: "Are you *sure* that it is impossible to trisect the angle by Euclid? I have not to lament a single hour thrown away on the attempt, but fancy that it is rather a tact, a feeling, than a

squaring of the circle. Although all of these are somewhat special in character, they have, as Hobson remarks, "one great advantage for the purposes of historical study, viz. that their complete history as scientific problems lies, in completed form, before us." Taking the last of these problems, Hobson continues, "we possess indications of its origin in remote antiquity, we are able to follow the lines on which the treatment of the problem proceeded and changed from age to age in accordance with the progressive development of general mathematical science, on which it exercised a noticeable reaction. We are able to see how the progress of endeavors towards a solution was affected by the intervention of some of the greatest mathematical thinkers that the world has seen, such as Archimedes, Huygens, Euler, and Hermite. Lastly, we know when and how the resources of modern mathematical science became sufficiently powerful to make possible that resolution of the problem, which, although negative, in that the impossibility of the problem subject to the implied restriction was proved, is far from being a mere negation, in that the true grounds of the impossibility have been set forth with a finality and completeness which is somewhat rare in the history of science."

"If the question be raised, why such an apparently special problem, as that of the quadrature of the circle, is deserving of the sustained interest which has attached to it, and which it still possesses, the answer is only to be found in a scrutiny of the history of the problem, and especially in the closeness of the connection of that history with the general history of mathematical science. It would be difficult to select another special problem, an account of the history of which would afford so good an opportunity of obtaining a glimpse of so many of the main phases of the development of general mathematics; and it is for this reason, even more than on account of the intrinsic interest of the problem, that I have

proof, which makes us think that the thing cannot be done. No doubt we are influenced by the cubic form of the algebraic equation. But would Gauss's inscription of the regular polygon of seventeen sides have seemed, a century ago, much less an impossible thing, by line and circle?"

De Morgan replied: "As to the trisection of the angle, Gauss's discovery increases my disbelief in its possibility. When $x^{17} - 1$ is separated into quadratic factors, we see how a construction by circles may tell. But, it being granted $ax^3 + bx^2 + cx + d$ is *not* separable into a real quadratic and a linear factor, I cannot imagine how a set of intersections of circles can possibly give no more or less than three distinct points." Graves' Life of Sir Wm. R. Hamilton, vol. 3 (1889), pp. 433–435.

selected it as appropriate for treatment in a short course of lectures."

Thus Professor Hobson gave six lectures at Cambridge University during the Easter term of last year, on the history of the problem of the quadrature of the circle. These lectures now appear in print in the volume before us.

It is 160 years since Montucla published anonymously his little book* on the history of this problem. In 1831, again in anonymous fashion, it was revised and corrected by Lacroix. Some sixty years later came Rudio's elaborate and excellent work.† Then there have been the Bändchen of Schubert‡ and of Beutel.§ Other historical surveys, not published in separate volumes, were written by T. Muir,‖ B. Calò,¶ Cantor,** T. Vahlen†† and D. E. Smith.‡‡

Professor Hobson's sketch is divided, as Rudio's, into four chapters: one a general account of the problem, and one for each of three historical periods "marked out by fundamentally distinct differences in respect of method, of immediate aims, and of equipment in the possession of intellectual tools."

Introductory to the general survey we are reminded that from the time of the Greeks down to our own day the problem has been very widely known outside of mathematical circles and many such people have occupied themselves in seeking a solution. In fact, as Schubert remarks, the Greeks had a special word to designate this species of activity, namely τετραγωνίζειν, which means to busy one's self with the quad-

* Histoire des recherches sur la Quadrature du cercle . . . avec une addition concernant les problemes de la duplication du cube et de la trisection de l'angle, Paris, 1754, pp. xliii+304+8 pls.

† Archimedes, Huygens, Lambert, Legendre. Vier Abhandlungen über Kreismessung . . . mit einer Übersicht über die Geschichte des Problemes von der Quadratur des Zirkels, Leipzig, 1892, pp. viii+166.

‡ H. Schubert, Die Quadratur des Kreises in berufenen und unberufenen Köpfen. In Virchow-Holtzendorffs Sammlung, Heft 67, Hamburg, 1889. Translated in *Monist*, Jan., 1891, vol. 1, pp. 197–228; reprinted in *Smithsonian Report* for 1890, pp. 97–120.

§ E. Beutel, Die Quadratur des Kreises (Mathematische Bibliothek, Nr. 12), Leipzig und Berlin, 1913, pp. 75.

‖ Encyclopaedia Britannica, ninth edition, article "Squaring of the circle," 1887; eleventh edition, article "Circle," 1910.

¶ In Enriques's Fragen derElementargeometrie, 2. Teil, Leipzig u. Berlin, 1911.

** Vorlesungen über Geschichte der Mathematik, vol. 1, 3. Aufl., 1907; vol. 2, 2. Aufl., 1900, reprinted 1913; vol. 3, 2. Aufl., 1901; vol. 4, 1908.

†† Konstruktionen und Approximationen, Leipzig u. Berlin, 1911.

‡‡ In Monographs on Topics of Modern Mathematics, ed. by J. W. A. Young, New York, 1911.

rature. In later day, among the scientific cranks with whom almost every scientific society has had to deal, is the insistent circle squarer amusingly dissected by De Morgan in his Budget of Paradoxes. So pestered were the members of the French Academy that as far back as 1775 they passed a resolution not to examine, from that time on, any so-called solutions of the quadrature of the circle that might be handed in.

On pages 4–10 Hobson sets forth the implications of the problem and notes that the fact was well known to the Greeks that the problems of the quadrature and rectification of the circle were really equivalent. He emphasizes assumptions and limitations of euclidean constructions and remarks, without direct reference to Mascheroni, that all such are possible with compasses alone. His discussion is then summed up: As an ideal problem the quadrature of the circle "is capable of accuracy bounded only by the instruments employed. Ideally we can actually determine, by euclidean methods, a square of which the area differs from that of a given circle by less than an arbitrarily prescribed magnitude, although we cannot pass to the limit."

The first historical period commences with the early empirical determinations of the ratio of the circumference of a circle to its diameter, and ends with the invention of the calculus in the latter part of the seventeenth century. This period was characterized by geometric discussion, for example, the method of exhaustions coupled with polygons circumscribed and inscribed. The first determination of the value of π is to be found in the Papyrus Rhind preserved in fragmentary condition in the British Museum and now being translated into English by Chancellor and Mrs. Chace of Brown University. This papyrus, written about 1700 B.C., is based upon a much earlier work. We here find $\pi = 256/81 = 3.1604 \cdots$. From the time of Chou-Kong, who lived in the twelfth century B.C., Chinese mathematicians employed the approximation $\pi = 3$. This value was also known to the Babylonians and is implied in verses of the Bible in Kings and Chronicles which are believed to have been written in the sixth and fifth centuries B.C. About 420 B.C. Hippias of Elis invented, for the solution of the problem, a curve known as the $\tau\epsilon\tau\rho\alpha\gamma\omega\nu\iota\zeta\text{-}o\upsilon\sigma\alpha$ or quadratrix, and about the same time Hippocrates of Chios developed his theory of menisci or lunulæ. But the first really scientific treatment was given by Archimedes

(287–212 B.C.). Considering circumscribed and inscribed polygons of 96 sides, he was led to the result $3\frac{1}{7} < \pi < 3\frac{10}{71}$ that is $3.14285\cdots < \pi < 3.14084\cdots$. The next more exact determination was found by Ptolemy (87–165 A.D.) to be $3\frac{17}{120} = 3.14166\cdots$. Four centuries later the great Chinese astronomer Tsu Ch'ung-chih proved that 10π lies between 31.415927 and 31.415926 and thence deduced the value $\frac{355}{113}$,* which is correct to six places of decimals. François Viète (1540–1603) obtained the value of π correct to 9 places, Adrianus Romanus to 15, Ludolph van Ceulen to 35. Huygens, Descartes, and Gregory introduced new methods of discussing the problem. Of approximate euclidean solutions may be mentioned that of J. de Gellert (*Grunert's Archiv*, 1849) which is readily deduced from the fact that $\frac{355}{113} = 3 + \frac{4^2}{7^2+8^2}$. The integraph, invented and described by the Russian engineer, Abdank-Abakanowicz, and its application to the construction of π, are nowhere mentioned by Hobson.

The second period, from the second half of the seventeenth to the latter part of the eighteenth century, is characterized by analytic discussion of the problem. J. Wallis, Lord Brouncker, Gregory, Machin, Leibnitz, Euler, and others obtained various expressions for π involving inverse tangents of numerical quantities, definite integrals, infinite products and series. Machin's relation

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

was used by Shanks who gave, in 1873, the value of π correct to 707 places of decimals. Professor Hobson dwells on Euler's work, for it is recalled that the final settlement of the problem was based upon the remarkable relation which he discovered,† namely

$$e^{i\pi} + 1 = 0,$$

* This is very easily remembered by writing down the first three odd numbers, each repeated as 113355, and recalling that the first three digits are to be placed in the denominator, the last three in the numerator.

† The form $e^{ix} = \cos x + i \sin x$ was first given by Euler in *Miscellanea Berolinensia*, vol. 7 (1743), p. 179; paper read Sept. 6, 1742. He used the symbol e in a letter dated November 25, 1731 (Corresp. Mat. et Phys., par Fuss, vol. 1 (1843), p. 58. He employed i for $\sqrt{-1}$ in 1777 (Institutionum calculi integralis, vol. 4 (1794), p. 184: "De formulis differentialibus . . .," presented to the Akademie, May 5, 1777). It should be noted

Although in this period great progress was made toward the solution of the problem, the true nature of the number π had not yet been discussed; but from the end of the seventeenth century none of the prominent mathematicians doubted that π was irrational.*

The third period commences with the proof of this fact by J. H. Lambert (1728–1777). Rudio's statement that this proof "fehlte zur völligen Strenge" has been copied without checking by Klein, Smith, and others, but Pringsheim showed that Rudio was in error. Following him, Vahlen and Hobson state the matter correctly. Nearly 80 years were to elapse before Liouville proved the existence of transcendental numbers. Hobson reproduces the 1851 presentation, in the main. Not till 1873 did Hermite prove the transcendence of e. Finally in 1882 Lindemann succeeded in establishing the transcendence of π and the problem whose history covers 4000 years was finally solved. Simplifications of Lindemann's proof soon followed, and one of these by Gordan is the basis of the condensed four-page Hobson proof, which requires knowledge of the elements of the differential calculus. It is assumed on page 55 that there are an infinite number of prime numbers. This was known to Euclid (Elements XI, 20.)

On page 50 the author establishes the theorem: "In order that a point P may be determinable by euclidean procedure it is necessary and sufficient that each of its coordinates be a root of an equation of some degree, a power of 2, of which the coefficients are rational functions of $(a_1, a_2, \cdots, a_{2r})$, the coordinates of the points given in the data of the problem." Hobson starts with

$$x = a + b\sqrt{c_1 \pm \sqrt{c_2 \pm \sqrt{c_3 + \cdots}}} + b'\sqrt{c_1' \pm \sqrt{c_2' \pm \sqrt{c_3' + \cdots + \cdots}}}. \tag{1}$$

But it is not shown that the equation referred to in the theorem may not have other roots than those given by (1). Multiple

that Roger Cotes gave the formula $ix = \log(\cos x + i \sin x)$ (*Phil. Trans.*, 1714), which is practically the same as the above as we look at it now, long before Euler.

* Euler, Introductio in Analysin Infinitorum, vol. 1 (1748), p. 93. It would be interesting to have Professor Hobson's authority for the statement that Euler gave expression to the conviction "that e and π are not roots of algebraic equations."

roots derived from (1) are not considered either, and the term "irreducible equation" is not mentioned. For this discussion as well as for the proof of the transcendence of π, the student will probably find more satisfactory and more elementary treatment in Klein's remarkable little book, Famous Problems in Elementary Geometry,* to which our author nowhere refers.

Professor Hobson's most interesting and attractively written book is such as to fire the imagination of a young student of mathematics and as to impel him to more persistent and enthusiastic effort. It is sure to have a wide circulation. So much the more then is it to be regretted that lamentable lack of care is displayed in the proofreading and final checking of details, by the author. His authorities, we are informed, included Heath,† Rudio, Cantor, Mikami,‡ and Vahlen, who can rarely, if ever, be blamed for the errors committed. Against an historical work coming from such a source, these are serious charges. It may therefore be well to support them at some length.

Obvious slips in typography occur on page 46, lines 3 and 10, page 55, line 18, in the exponent of C. On page 55, lines 25, 27, 30, with usual notation for differentials, brackets should surround the exponents of ϕ. In mathematical writing, uniformity seems highly desirable, at least within the limits of a 60 page book: why then the proper names in italics on pages 23, 25, 27, and why the three forms Leonardo Pisano, Nicholas of Cusa, Leonardo da Vinci? Three different methods are used for referring to the volume of a periodical work (for example, page 31, and page 44, lines 6 and 9) while one of these methods is elsewhere used for a book in a volume (page 31, line 29).

Before the abbreviation "Kl." on page 44 and again in a footnote on page 53, "Math.-nat." should be placed. Many names are spelled incorrectly: In seven places an extra *h* is put in Huygens, Raaus should be Ra-ā-us', for Áryabhatta read Âryabhaṭṭa; for Chang Hîng, Chang Hêng; for Wang Fau, Wang Fan; for Gancea, Ganeça; for Alchwarizmi, Alchwarizmî; for George Purbach, Georg von Peurbach

* Boston, 1897; German ed., Leipzig, 1895; French ed., Paris, 1896; Italian ed., Torino, 1896; Russian ed., Kasan, 1898.

† Works of Archimedes, Cambridge, 1897.

‡ The Development of Mathematics in China and Japan, Leipzig, 1912.

or Georg of Peurbach; for Rheticus, Rhäticus; for Ludolf van Ceulen, Ludolph van Ceulen; for Willebrod Snellius, Willebrord Snellius; for Grünert, Grunert; for Tanyem Shŏkei, Tanzan Shōkei.

We are told that Professor Landau discussed lunulæ in 1890, that is when he was 13 years of age. This date should be 1902, and the reference to "Ueber quadrierbare Kreisbogenzweiecke," *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, volume 2 (1902), pages 1–6. Three dates of birth are given incorrectly: Ludolph van Ceulen, 1539 instead of 1540 (N.S.); Newton, 1642 instead of 1643 (N.S.); Snellius, 1580 instead of 1581. The phrase "is said to have," page 27, line 4, should be deleted and after the next sentence, "According to his wish the value was engraved on his tombstone which has been lost," should come: "but the inscription has been preserved (Bierens de Haan, *Messenger of Mathematics*, volume 3 (1874), page 25)."

Vega calculated π correctly to 136 not 140 places.* Two slips occur in connection with W. Shanks' name (page 39): (1) he published his value of π to 530 places in *Proceedings of the Royal Society*, 1853–54, but his value to 607 places in Contributions to Mathematics, London, 1853, pages 86–87; (2) he gave 707 places in 1873, not "1873–74." On page 40 for "1766" read "1769," the year when the result was printed; and on page 39 for "1755" read "1779 (though not published till 1798);"† for "The same series was also discovered independently by Ch. Hutton" read "The same series was first discovered by Ch. Hutton."

The statement on page 41: "He [Euler] introduced the practice of denoting each of the sides and angles of a triangle by a single letter," is inaccurate, as Caswell‡ (and Oughtred) had already named each of the sides by a letter. Hobson might have correctly written: "Euler was one of the first if not the first to introduce the notation a, b, c, for the sides

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* G. Vega, Thesaurus Logarithmorum, Leipzig, 1794, p. 633, and W. Shanks, Contributions to Mathematics, London, 1853, p. 86.

† "Conventuri exhibita die 7 Junii, 1779" *Nova Acta Acad. sc. Petrop.*, vol. 11 (1793), 1798, p. 133. Cf. G. Enestrŏm, Verzeichnis der Schriften Leonhard Eulers, Leipzig, 1910, p. 181.

‡ "Suppose m, n the legs of the angle required; B its base . . . $\frac{1}{2}$ sum m, n, B," page 6 of A Brief (but full) Account of the Doctrine of Trigonometry Both Plane and Spherical, by John Caswell, London, 1685; bound in with Wallis's Treatise of Algebra, London, 1685.

opposite the angles A, B, C, of a triangle, etc."* And again, did Euler introduce "the short designation of the trigonometrical ratios by $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, etc."? The originals of the remarkable memoirs of 1753 and 1781† are not available to me, but in E. Hammer's translations‡ "tan" never occurs; it is always "tg"; so also "ctg" not "cot"; furthermore the abbreviation "cosec" or "csc" does not come in at all. In 1777 J. T. Mayer used the following abbreviations:§ sin, cos, tang, cot, sec, cosec.

Professor Hobson recalls that "the notation π appears as early as 1706, when it was used by William Jones in Synopsis palmariorum matheseos." It would have been of interest to have further remarked that the notation was probably suggested to Jones by Oughtred‖, whose name stands out much more prominently in English mathematics. Liouville's papers on transcendental numbers are dated 1844¶ and 1851,** not 1840 (page 44). To the sentence: "The simpler of Liouville's methods of proving the existence of such numbers will be given here" should be added, "with some alterations." And finally Professor Hobson seems to have forgotten (page 53) that it was the third, not the second, edition of his Treatise on Trigonometry which was published in 1911.

In broad outline the work is accurate; but when the mathematician writes now-a-days, he is expected to have a much higher ideal. As clearly stated in the preface, no original contribution to the history of the subject is attempted; for this we must await the forthcoming edition of Pappus by

* This was done in the case of *spherical* triangles in 1753: "Principes de la trigonométrie sphérique tirés de la méthode des plus grands et plus petits," *Mém. de l'Acad. d. Sc. de Berlin*, vol. 9 (1753), 1755, pp. 223–257.

† *Acta Acad. Sc. Petrop.*, 1779; I, printed 1782, pp. 72–86. "Nach den Akten am 12 März, 1781 der Petersburg Akademie vorgelegt." (G. Eneström, Verzeichnis der Schriften L. Eulers, Leipzig, 1910, p. 149.)

‡ Ostwald's Klassiker der exakten Wissenschaften, No. 73, Leipzig, 1896.

§ Gründlicher und ausführlicher Unterricht zur praktischen Geometrie, Erster Theil, Göttingen, 1777, p. 5 ff.

‖ William Oughtred (1574–1660) in his Clavis Mathematica of 1631, etc., and in his Theorematum in libris Archimedes de Sphaera et Cylindro Declaratio, Oxford, 1652, frequently employs the symbol $\delta:\pi$ or $\pi:\delta$ (in modern rotation) for the ratio of the semi-diameter to the semi-periphery or of semi-periphery to semi-diameter. It is noticeable that these letters are *never* used separately, that is, π is *not* used for "Semiperipheria," as Tropfke suggests (Geschichte der Elementar-Mathematik, vol. 2 (1903), p. 135). Oughtred states specifically in his "Theorematum": "$\pi R/\delta$, est semiperipheria circuli cujus Radius est R."

¶ *Comptes Rendus*, vol. 18, pp. 883, 910.

** *Liouville's Journal*, vol. 16, pp. 134, 139.

Professor Hobson's fellow countryman, Sir Thomas L. Heath. It is certainly not well to overload a work of this kind and suggestions for filling up the many lacunae are hardly in order. It appears to the reviewer, however, that the value and usefulness of the work would be considerably enhanced if many more exact references were given to original sources as authorities for the statements that are made, and if an index of names were appended.

In conclusion, and in illustration of the statement made above, that others beside mathematicians were cognizant of the problem of squaring the circle, it may be of interest to put on record two references to literary classics. In the Birds (produced in 414 B.C.) of Aristophanes there is a long dialogue between Peisthetaeros and Meton, the great astronomer, introduced as a solemn quack talking unintelligible nonsense for the most part. In lines 1004–5, however, Meton is made to say:*

> "With the straight rod I measure out, that so
> The circle may be squared."

The second reference takes us to the early fourteenth century. It is to the last canto of the Paradiso where Dante compares his inability to penetrate by his own unaided power the mystery of the Incarnation, to that of one

> "Who versed in geometric lore, would fain
> Measure the circle;† and though pondering long
> And deeply, that beginning, which he needs,
> Finds not; . . . ."‡

Are there references to other passages of this kind before 1500? One may turn to the great Oxford Dictionary for those of later date.

R. C. ARCHIBALD.

BROWN UNIVERSITY,
PROVIDENCE, R. I.,
January 30, 1914.

* Translation of B. B. Rogers, London, 1906.
† "Misurar lo cerchio;" Longfellow translates this "To square the circle."
‡ Cary's translation, Canto 33, lines 133–135.

SHORTER NOTICES.

New Analytic Geometry. By PERCY F. SMITH and ARTHUR SULLIVAN GALE. Boston, Ginn and Company, 1912. x+342 pp.

AT the present time when there are so many text-books which treat of analytic geometry, a new book in order to justify its publication should be original both in its methods and in the material which it contains. If this is taken as our standard, Smith and Gale's New Analytic Geometry ought certainly to find a place among the modern college text-books of elementary mathematics. The authors have succeeded in getting away from the beaten and orthodox path followed by a majority of similar text-books and have given to the subject a treatment which is in harmony with progressive ideals in college mathematics.

In the selection of material the book is unique, if we except with some reservation a few other books such as Lambert's Analytic Geometry or Smith and Gale's Introductory Course in Analytic Geometry. In the table of contents are to be seen such chapter headings as Transcendental curves and equations, Functions and graphs, Parametric equations and loci, Empirical equations, etc. A hasty glance is sufficient to give the impression that the Analytic Geometry is not a treatise on conic sections, the discussion of that important class of curves being limited to one short chapter of thirty-five pages. The authors have emphasized the fact that they are presenting a general treatment of loci problems, and that corresponding equations can be determined and interpreted regardless of their forms or of the kinds of coordinates which are used. The aim seems to have been not to study intensively a limited class of curves and equations but rather to develop general laws which can be applied equally well to all parts of the field.

The teacher of calculus and of more advanced mathematics is naturally pleased to see the new chapters on transcendental curves and parametric equations as well as an excellent chapter on polar coordinates. In these parts of the book are found a great many of the typical equations and curves which are used in illustrations in calculus and which appear very frequently in technical work. More space has been given to

transcendental and parametric equations than to the equations of the conic sections. It must, however, be admitted that, from the standpoint of the student of technology, this division of space can be justified.

A chapter on Empirical equations is, for a text-book of analytical geometry, decidedly an innovation. However, the treatment is far from adequate and lacks almost completely any evidence of mathematical rigor. Such a chapter, however, may be the means of stimulating the reader to investigate more carefully this important field of applied mathematics. It is exceeding doubtful whether any treatment of the subject of curve fitting can be so condensed and simplified as to find a place in a text-book of analytic geometry and at the same time be adequate from a strictly mathematical point of view.

Ninety-seven pages of the text are given over to the analytical geometry of space. This part of the book, however, seems to lack much of the originality and freshness which characterizes the rest of the book. If the treatment is superior to the same parts of other text-books it is due to the most excellent list of problems which are included.

In their presentation of the subject the authors have used a modified form of the euclidean method. The important theorems are formally stated and the formal proof ending with the usual Q. E. D. follows. Among these important theorems are found those for the distance between two fixed points, the slope-intercept equation of the straight line, the angle between two lines, the equation of a circle, etc. Other theorems are presented as problems. The problem is stated and the solution given, then the result of the solution is summarized in a theorem or in a set of general rules to be used in solving specific problems of the same general character. A third method of presentation is also used which makes use of an illustration. For example, the method of deriving the equation of the tangent at any point on a curve (see page 190) is developed by means of a solution of the problem for the cubical parabola. While this illustrative method may be criticized because of its lack of rigor and generality, it is certainly quite often much more intelligible to the average student than a long and abstract presentation of the same thing. By using these three ways of presentation the strict euclidean method, which was used in the former treatise by the same authors, has lost much of its harshness.

The mechanical side of the book has been splendidly executed and is almost entirely free from errors. The use of italics and heavy type is not overdone, but is still sufficient to emphasize the really important things of the text. The figures are abundant and well drawn. The problems are numerous and have been selected because they illustrate something, rather than because of their difficulties.

Taken as a whole, the book is thoroughly up to date and well written. Although it is more of a drill book than a treatise, it ought nevertheless to furnish the student with a good foundation for a later course in calculus. It is a book which should be a stimulus to every teacher who is in sympathy with the international movement towards improvement in the curriculum and the methods of instruction in college mathematics.

EDWIN R. SMITH.

Technical Trigonometry. By HORACE WILMER MARSH. New York, John Wiley and Sons, 1914. x+232 pp.

THIS book gives a clear and usable knowledge of the trigonometry underlying the industrial and technical studies. The first chapter is devoted to an explanation of logarithms, while the second and third chapters are given to the solution of right and oblique triangles with their applications. It is in the applications that this book differs from the traditional trigonometries. The exercises are chosen from engineering, physics, manufacturing, etc. A few of the types of problems given are: equilibrium on an inclined plane, bevel gears, spiral gear cutter, dovetail joints, two point ball bearing, nuts, flange angles, tangent galvanometer, track turnout, roof truss, length of belts, concrete stand pipes, sewer construction. Each problem is accompanied by a very well executed drawing. These three chapters cover 186 pages or about 9/11 of the book. Many technical terms are used and explained.

The fourth chapter (13 pages) deals with the functions of the sum and difference of two angles and the functions of multiple angles. The proofs of most of the formulas are left as exercises. No other exercises such as proving identities, solving trigonometric equations, solving practical problems, etc., are given. In the last chapter, namely chapter five, the slide rule is explained. At the end of the book are found tables on length, area, volume, weight, decimal equivalents of

parts of an inch, U. S. and metric equivalents and wire gauge sizes.

No tables of logarithms or natural functions are given. Radian measure, identities, trigonometric equations and answers to the exercises have been omitted.

F. M. MORGAN.

College Algebra. By W. B. FITE. Boston, Heath, 1913.

A CHARACTERISTIC tendency in mathematics during the last fifty years or more has been in the direction of a more severe criticism of the proofs of theorems. Hence, Professor Fite is adopting the modern point of view when he aims "to present the elementary principles of algebra in a simple and direct way and to give a rigorous proof of the theorems used." But just at this point one of the real dangers in the teaching of elementary mathematics arises. Those who are interested in emphasizing rigor are likely to be sidetracked by minutiæ to such an extent that the subject as presented by them appears too abstract to the beginner, to whom concreteness in any subject is an absolute necessity. With these facts in mind one might say that the aim of the modern text book in algebra should be to present the subject as concretely as possible and at the same time to develop the critical attitude in the mind of the student. I shall examine Professor Fite's College Algebra chiefly along these lines.

First of all it should be noted that the book is meant to be a *college* algebra. Although all of the subjects treated occur in most of the texts which are used in secondary schools it seems to me that some of the subjects, notably the analytic geometry introduced, belong exclusively to college work. This does not mean that the graph and graphical methods should not be used in secondary work; they should be, by all means. Also, the problems which have been selected are largely new and very attractive, and are of sufficient difficulty to make the average freshman think.

Throughout the book the author has made extensive use of graphical methods and has evidently tried to appeal to intuition and to make the subject tangible by practical illustrations of the principles involved. To this end the problem of finding the equation of a circle through three points is introduced as an application of the solution of simultaneous linear equations in three variables. This is better than to refer to three planes,

which is always somewhat unsatisfactory at this period of study. A great many more problems from geometry appear than one usually finds in such texts. The same might be said of physical problems, for which an appendix has been added concisely explaining the fundamental laws of physics. The author does not hesitate to state facts which can be ascertained only by means of higher mathematics to be used as the basis of an algebraic problem; this, to my mind, is an excellent method of encouraging the thoughtful student to pursue the study of mathematics further. To the student who can handle the formal work of the book and understand the problems, algebra should be a tool by means of which certain kinds of information might be obtained.

The importance of rigor is never forgotten. In an early chapter the operations which may be legitimately used in the solution of equations are stated and illustrated. The chapter on mathematical induction—a difficult topic to teach successfully—is admirable in this respect. In the proof of the remainder theorem in §142 that the two sides of the expression are *identically* equal is carefully noted; some texts seem to forget the importance of this step. Another evidence of this spirit of precision appears at the bottom of page 177 and need only be mentioned here. Several proofs are very properly omitted; for instance, the proof that every polynomial in x has at least one root is merely referred to historically; the proof of De Moivre's theorem is left to the student; but all theorems which are proved at all are *rigorously* proved, and wherever it is necessary critical observations are added.

Among other features of the book may be mentioned the chapter on complex numbers. The presentation of this subject is somewhat new and very attractive. The chapter on Inequalities contains much which is useful and which does not appear in any other modern text with which I am familiar. In the proof of theorems by mathematical induction illustrations are given which show convincingly the necessity of both parts of the proof.

I have the following adverse criticism to offer. An exceedingly long and complicated sentence for a text book occurs at the bottom of page 22, the purpose of which could have been accomplished better by several more illustrative examples. The subject of undetermined coefficients could have been briefly explained; the omission of this powerful instrument of

algebra to make place for so much geometry is hardly justifiable in a text of this kind. Possibly it would have been of advantage to pay more attention to the theory of the quadratic equation—a subject which so many freshmen have not mastered. On page 155, line 4, the word "inequalities" should be changed to "equalities."

Finally, it is my opinion that the book is distinctly a college text; that the subject has been concretely presented, and that an effort has been made to awaken the spirit of critical revision in the mind of the student.

JOSEPH EUGENE ROWE.

Das Problem der Kreisteilung. Ein Beitrag zur Geschichte seiner Entwicklung. Von DR. ARTHUR MITZSCHERLING. Leipzig, Teubner, 1913. vi + 214 pp. M 7.

THIS book falls naturally into three sections, treating the following topics: (*a*) the division of the circle into equal parts, (*b*) the trisection of angles, (*c*) the polysection of angles. Each section is written from the historical point of view and contains for its general topic an account of the relevant geometric constructions, both exact and approximate, and of instruments by means of which the corresponding practical constructions may readily be made.

The book will be of interest to those who desire an elementary historical account of the topics treated.

R. D. CARMICHAEL.

Allgemeine Theorie der Raumkurven und Flächen. Von V. KOMMERELL und K. KOMMERELL. (Sammlung Schubert XXIX and XLIV.) Zweite Auflage, I. Band, 1909, pp. viii+172; II. Band, 1911, pp. vi+188. G. J. Göschen'sche Verlagshandlung, Leipzig. Price 4.80+5.80 marks.

WITHIN the last two or three years, new works, or old works in new editions, have accumulated rapidly in the field of differential geometry. For do we not have Bell, Darboux, Demartres, Forsyth, Knoblauch, Kommerell, Lilienthal, Salmon, Scheffers, and Smith? There are many American colleges, no doubt, where lecture courses, and texts in a foreign language, are deemed inadvisable. In such places the teacher who offers an introductory course in differential geometry is distinctly at a loss. Bell, Frost, Salmon, and

Smith will not, probably, appeal to any American professor as suitable texts in this connection, whilst the exposition of Eisenhart and of Forsyth is too detailed or is otherwise inadequate. Indeed, only one suitable work is available. That is Goursat-Hedrick's Analysis, volume I.* With portions of this work somewhat amplified, with the English edition of Gauss's famous memoir published by the Princeton University Library, and with selections from Smith, a satisfactory course may be worked up. An appropriate single volume in English is therefore much to be desired. We need such elementary books as Raffy, with its clear-cut, elegant and rigorous demonstrations, or this new edition of Kommerell with theory presented in simple form interspersed with ample problem illustration, while authoritative comment† on the historical development stimulates the interest.

The new edition of Kommerell differs widely from the old, which appeared in 1903. There are corrections, freshly drawn figures, expansions, new paragraphs, rearrangements, new illustrations, new problems, and the elaboration of the second volume of 218 pages into one of 194 and another of 177. The original work of 350 pages has been increased in size by 200 pages, to form a treatise of about double the size of Raffy.

To the first volume only 28 pages were added and, as before, it is devoted to the discussion of space curves, and of surfaces whose equations are in the form $F(x, y, z) = 0$. The second volume originally contained two sections (each of which is now treated in a separate volume): the first (118 pages) on surfaces whose equations are in parametric form, the second (94 pages) on special surfaces (W, minimal, of constant curvature, ruled, triply orthogonal) and congruences.‡

For use in the field indicated, the volumes can be heartily recommended.

R. C. ARCHIBALD.

* Boston, 1904. A new edition in two volumes may be expected in a year or so.

† Dr. V. Kommerell was the author of "Analytische Geometrie der Ebene und des Raumes" in Cantor's Vorlesungen über Geschichte der Mathematik, Bd. 4 (1908), pp. 453–576.

‡ The new volume on this subject is entitled: Spezielle Flächen und Theorie der Strahlensysteme, Leipzig, 1911, and was reviewed in this BULLETIN, vol. 19 (1913), pp. 253–4, by Professor Cowley.

Beiträge zu einzelnen Fragen der höheren Potentialtheorie. Von E. R. NEUMANN. (Preisschriften gekrönt und herausgegeben von der Fürstlich Jablonowskischen Gesellschaft zu Leipzig, No. 41.) Leipzig, Teubner, 1912. xxiv+188 pp.

THIS prize memoir is concerned with the two fundamental problems of the potential theory, viz., the determination of a solution V of Laplace's equation satisfying certain continuity conditions either in the domain inside or in that outside the boundary when, for every boundary point s, either the value of V or the derivative $\partial V/\partial n_s$ along the boundary normal n_s is given (first and second boundary problem, respectively).

For the first problem, the method of Green's function gives

$$V_p = \int f_s \psi_{sp} ds, \tag{1}$$

where f_s is the given boundary value, and ψ_{sp} a function depending on the boundary point s and the variable point p only. The researches of Carl Neumann have suggested the possibility of representing ψ_{sp} as the potential of a mass distribution *on the boundary*

$$\psi_{sp} = \int \Delta_{s\sigma} T_{\sigma p} d\sigma, \tag{2}$$

where $T_{\sigma p} = 1/r_{\sigma p}$ for the three-dimensional and $= \log r_{\sigma p}$ for the two-dimensional problem, and $\Delta_{s\sigma}$ depends only on the two boundary points s and σ. The investigation of this "fundamental mass distribution" $\Delta_{s\sigma}$ is the main object of the memoir under review.

The method is briefly the following: expressing ψ_{sp} by the infinite series furnished by Carl Neumann's method for solving the first boundary problem, this series is divided into two parts. The first consists of a finite number of terms and contains all the singularities of ψ_{sp}, while the second part, consisting of the remainder of the series, has no singularities. It is found that the representation (2) is possible for the second part, but not for the first, which, however, is of sufficient simplicity in itself.

Making $h = 2$ for the three-dimensional and $h = 1$ for the two-dimensional problem, the following formulas are obtained, according as the variable point p is an interior point i or an exterior point e:

$$\psi_{si} = \frac{1}{h\pi}(\gamma_{si} - \gamma_{si}') + \int \Delta_{s\sigma} T_{\sigma i} d\sigma,$$

$$\psi_{se} = \frac{-1}{h\pi}(\gamma_{se} + \gamma_{se}') + \int \Delta_{s\sigma} T_{\sigma e} d\sigma, \tag{3}$$

where in both cases

$$\gamma_{sp} = \frac{\partial}{\partial n_s} T_{sp}, \quad \gamma'_{sp} = \frac{1}{h\pi} \int \gamma_{\sigma p} \frac{\partial}{\partial n_s} T_{s\sigma} d\sigma. \tag{4}$$

A similar investigation is carried through for the second boundary problem; here the result is somewhat simpler. The author now proceeds to investigate the relation between the fundamental distributions on a surface and its transform by reciprocal radii, and furthermore gives a detailed discussion of the special cases of the circle, sphere, and ellipse. Certain convergence investigations concerning the Neumann-Robin methods conclude the memoir.

T. H. GRONWALL.

Physical and chemical Constants and some mathematical Functions. By G. W. C. KAYE, the National Physical Laboratory, England, and T. H. LABY, professor of physics, Wellington, N. Z. London, Longmans, Green, and Co., 1911. 153 pp.

THE authors have written this book to supply the need for a set of up-to-date English physical and chemical tables of convenient size and moderate price. In comparison with larger works such as those of Landolt, Bornstein, and Meyerhoffer this little book contains more information than its relative size indicates. In fact for class use where one illustration of a given type is good as another this book is ample. For the investigator desiring information of a very particular nature it may not be sufficient.

In most cases a table is accompanied by a brief statement containing definitions and standard formulas. Many references are given both to standard works and to observers for particular measurements. Interpolations and extrapolations are indicated.

The book contains several hundred tables covering subjects in general physics, astronomy, heat, sound, light, electricity,

magnetism, radioactivity and gaseous ionization, and chemistry. At the end are tables of e^{-x}, four and five-place logarithms, reciprocals, squares, and natural trigonometric functions.

H. B. PHILLIPS.

NOTES.

THE annual list of American doctorates published in *Science* presents, for the academic year 1913–1914, 502 names, of which 241 are credited to the sciences. The following 27 successful candidates offered mathematics as major subject (the titles of the theses are appended): E. S. ALLEN, Harvard, "Su alcuni caraterri di una serie algebrica, e la formola di de Jonquières per serie qualsiasi;" R. D. BEETLE, Princeton, "Congruences associated with a one-parameter family of curves;" Miss S. R. BENEDICT, Michigan, "A comparative study of the early treatises introducing into Europe the Hindu art of reckoning;" B. A. BERNSTEIN, California, "A complete set of postulates for the logic of classes expressed in terms of the operation 'exception' and a proof of the independence of a set of postulates due to Del Ré;" J. A. BULLARD, Clark, "On the structure of finite continuous groups;" R. W. BURGESS, Cornell, "The uniform motion of a sphere through a viscous liquid;" Miss L. D. CUMMINGS, Bryn Mawr, "On a method of comparison for triple-systems;" J. W. GAIN, Johns Hopkins, "Linear combinants of ternary forms;" H. GALAJIKIAN, Princeton, "A type of non-linear integral equations;" H. C. GOSSARD, Johns Hopkins, "On a special elliptic ruled surface of the ninth order;" G. H. GRAVES, Columbia, "Complete linear systems of algebraic curves of least order of genera three and four;" L. M. KELLS, Columbia, "Complete characterization of dynamical trajectories in n-space;" E. A. T. KIRCHER, Illinois, "Group properties of the residue classes of certain Kronecker modular systems and some related generalizations in number theory;" W. C. KRATHWOHL, Chicago, "Modular invariants of two pairs of cogredient variables;" T. E. MASON, Indiana, "Character of the solutions of certain functional equations;" L. C. MATHEWSON, Illinois, "Theorems on the groups of isomorphisms of certain groups;"

Miss B. I. MILLER, Johns Hopkins, "A new canonical form of the elliptic integral;" L. W. MISER, Chicago, "On linear homogeneous differential equations with elliptic function coefficients;" F. M. MORRISON, Chicago, "On the relation between some important notions of projective and metrical differential geometry;" E. J. MOULTON, Chicago, "On figures of equilibrium of a rotating heterogeneous fluid body;" R. B. ROBBINS, Harvard, "The calculus of variations as the limit of a problem in minimizing an algebraic sum;" W. F. SHENTON, Johns Hopkins, "Linear combinants of systems of binary forms with the syzygies of the second degree connecting them;" S. P. SHUGERT, Pennsylvania, "König's resolvents and other types of symmetric functions;" J. M. STETSON, Princeton, "Conjugate systems both of whose Laplace transforms are lines of curvature;" Miss A. H. TAPPAN, Cornell, "Plane sextic curves invariant under birational transformations;" L. A. H. WARREN, Chicago, "A class of asymptotic orbits in the problem of three bodies;" Miss M. M. YOUNG, Johns Hopkins, "Dupin's cyclide as a self-dual surface."

THE course of lectures on General analysis to be given by Professor M. FRÉCHET at University of Illinois has been cancelled on account of the war.

THE Berlin Academy of Science has awarded Dr. P. KOEBE, of Leipzig, a prize of 5,000 marks for his work on the theory of functions.

THE Paris Academy of Sciences has granted 4,000 francs from the Bonaparte fund to Professor H. ANDOYER to assist in the publication of a set of trigonometric tables.

PROFESSOR KARL RUNGE, of the University of Göttingen, has been chosen Prorektor for the coming year.

IN honor of his holding the doctorate fifty years, the Technical School at Munich has conferred its honorary doctorate in engineering on Professor A. VON BRILL of Tübingen.

DR. A. N. WHITEHEAD of University College, University of London, has been appointed to the newly established chair of

applied mathematics in the Imperial College of Science and Technology of London. Mr. G. N. WATSON, fellow of Trinity College, Cambridge, has been appointed to the staff of the University College in succession to Dr. Whitehead.

PROFESSOR G. T. ROGERS, of the New Mexico Normal School, has been elected assistant dean and professor of mathematics in the Normal University of New Mexico.

AT the University of California, Dr. C. E. BROOKS has been appointed assistant professor of mathematics and insurance, and Professor T. M. PUTNAM has been promoted to an associate professorship of mathematics.

PROFESSOR L. L. DINES, of the University of Arizona, has been appointed assistant professor of mathematics in the University of Saskatchewan.

AT the University of Kansas, Professor J. N. VAN DER VRIES has been promoted to a full professorship of mathematics. Dr. E. B. STOUFFER, of the University of Illinois, has been appointed assistant professor of mathematics, and Mr. E. H. CARUS instructor in mathematics. Professor Marion B. WHITE has resigned, to accept an appointment as associate professor of mathematics and dean of women at the Michigan state normal college at Ypsilanti.

AT the University of Michigan Dr. W. W. KÜSTERMANN and Mr. V. H. WELLS have been appointed instructors in mathematics.

DR. W. C. GRAUSTEIN, of Harvard University, has been appointed instructor in mathematics in the Rice Institute, Houston, Texas.

DR. F. B. WILEY has been appointed professor of mathematics in Denison University.

DR. J. L. JONES has been appointed instructor in mathematics in the University of Pittsburgh.

DR. M. G. GABA has been appointed instructor in mathematics in the Carnegie institute of technology at Pittsburgh.

DR. J. M. STETSON has been appointed instructor in mathematics in the University of Alberta.

PROFESSOR W. H. GARRETT, of Baker University, has received a year's leave of absence, which he will spend at Harvard University as student instructor.

MISS M. DANIELS and Mr. H. LATIMER have been appointed instructors in mathematics at the Iowa state college.

MISS M. A. COLPITTS has been appointed instructor in mathematics at Tabor College.

DR. W. J. MILNE, president of the Albany normal college, died September 4 at the age of 71 years. Dr. Milne had been a member of the American Mathematical Society since 1906.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ANTOMARI (X.). Cours de géométrie descriptive. Paris, Vuibert, 1914. 8vo. 641 pp. Relié. Fr. 12.50

APPELL (P.). See ENCYCLOPÉDIE.

ARCAIS (F. DI). Analisi infinitesimale. 2 volumes. Padova, 1913. L. 30.00

BACHMANN (P.). See KLEIN (F.).

BACKES (W.). Weitere Beweise für den Fermatschen Satz. 2te umgearbeitete und vermehrte Auflage von "Ein Beweis für den Fermatschen Satz." Mainz, 1914. 8vo. 18 pp. M. 3.00

BARONI (E.). See QUESTIONI.

BARTHEL (E.). Vertikaldimension und Weltraum. Leipzig, Hillmann, 1914. 28 pp. M. 0.75

BERNOULLI (J.). Die erste Integralrechnung. Auswahl aus den mathematischen Vorlesungen über die Methode der Integrale und anderes (1691–92). Aus dem Lateinischen übersetzt und herausgegeben von G. Kowalewski. Berlin, 1914. 8vo. 8+285 pp. M. 5.00

BÔCHER (M.). An introduction to the study of integral equations. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 10.) 2d edition. Cambridge, University Press, 1914. 8vo. 71 pp. 2s. 6d.

BOREL (E.). Le hasard. Paris, Alcan, 1914. (Nouvelle collection scientifique; directeur, E. Borel.) 12mo. 4+312 pp. Fr. 3.50

——. See NIEWENGLOWSKI (B.).

BOUTROUX (P.). See VOLTERRA (V.).

BRAUDE (L.). Les coordonnées intrinsèques. Théorie et applications. (Collection Scientia, No. 34.) Paris, Gauthier-Villars, 1914. 8vo. 100 pp. Fr. 2.00

BRENDEL (M.). See KLEIN (F.).

BURALI-FORTI (C.) e MARCOLONGO (R.). Analyse vectorielle générale, Volume II: Applications à la mécanique et à la physique. Pavia, Mattei, 1913. 8vo. 12+143 pp. L. 5.00

DANIELE (—.). See QUESTIONI.

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Tome II, volume 5, fascicule 2: Fonctions sphériques, exposé d'après A. Wangerin par A. Lambert, avec une note de P. Appell et A. Lambert, pp. 161–230; Généralisations diverses des fonctions sphériques, exposé par P. Appell et A. Lambert, pp. 231–268. Paris, Gauthier-Villars, 1914. 8vo. Fr. 4.80

ENRIQUES (F.). Problems of science. Authorized translation by Katherine Royce, with an introduction by Josiah Royce. Chicago, Open Court, 1914. 8vo. 16+392 pp. Cloth. $2.50

——. See QUESTIONI.

FORSYTH (A. R.). Lectures introductory to the theory of functions of two complex variables. Cambridge, University Press, 1914. 8vo. 16+281 pp. 10s.

FORT (O.) und SCHLÖMILCH (O.). Lehrbuch der analytischen Geometrie. In 2 Teilen. II. Teil: Analytische Geometrie des Raumes. 7. Auflage, bearbeitet von R. Heger. Leipzig, Teubner, 1913. 8vo. 8+326 pp. Cloth. M. 6.80

GAZZANIGA (P. E.). Nuovo ed importante teorema di geometria e sue conseguenze: geometria nuova, memoria. Piacenza, Del-Maino, 1914. 8vo. 35 pp. L. 1.00

GIBSON (G. A.). Napier and the invention of logarithms. (*Tr. Royal Philosophical Society of Glasgow.*) Glasgow, Carter and Pratt, 1914. 8vo. 24 pp.

GOURSAT (E.). Cours d'analyse mathématique. Tome III, 1er et 2e fascicules. Paris, Gauthier-Villars, 1913–1914. 8vo. 544 pp. Complete volume to cost subscribers Fr. 20.00

HADAMARD (J.). See VOLTERRA (V.).

HALL (T. P.). A geometrical vector algebra. Vancouver, B. C., Western Specialty Co., 1914. 8vo. 2+30 pp.

HAUSDORFF (F.). Grundzüge der Mengenlehre. Leipzig, Veit, 1914. 8vo. 8+476 pp. M. 18.00

HEGER (R.). See FORT (O.).

HOBSON (E. W.). John Napier and the invention of logarithms. Cambridge, University Press, 1914. 8vo. 48 pp. Cloth. 1s. 6d.

KENNELLY (A. E.). Chart atlas of complex hyperbolic and circular functions. Cambridge, Mass., Harvard University Press, 1914. Folio. 23 charts. $4.00

KLEIN (F.) und BRENDEL (M.). Materialen für eine wissenschaftliche Biographie von Gauss. Gesammelt von F. Klein und M. Brendel. Heft I: Ueber Gauss' zahlentheoretische Arbeiten, von P. Bachmann. Leipzig, Teubner, 1911. 8vo. 2+144 pp. M. 2.00

——. Materialen für eine wissenschaftliche Biographie von Gauss. Heft II: C. F. Gauss, Fragmente zur Theorie des arithmetisch-geometrischen Mittels aus den Jahren 1797–1799. Heft III: Ueber Gauss' Arbeiten zur Funktionentheorie, von L. Schlesinger. Leipzig, Teubner, 1912. 8vo. 2+144 pp. M. 4.40

KÖNIG (J.). Neue Grundlagen der Logik, Arithmetik und Mengenlehre. Leipzig, Veit, 1914. 8vo. 8+259 pp. M. 8.00

KOWALEWSKI (G.). See BERNOULLI (J.).

LAISANT (C. A.). Mathematics. New York, Doubleday, 1914. $0.50

LAMBERT (A.). See ENCYCLOPÉDIE.

LANGEVIN (P.). See VOLTERRA (V.).

LÉVY (P.). Sur les équations intégro-différentielles définissant des fonctions de lignes. (Thèse.) Paris, Gauthier-Villars, 1914. 4to. 4+122 pp. Fr. 6.00

LINDEMANN (F. und L.). See POINCARÉ (H.).

LOEBER (K.). Beiträge zur Lösung und Geschichte des Malfattischen Problems und seinen Erweiterungen. (Diss. Halle.) Berlin, H. Blankes Buchdruckerei, 1914. 8vo. 8+63 pp.+3 plates. M. 2.00

LONGLEY (W. R.). Tables and formulas for solving numerical problems in analytical geometry, calculus and applied mathematics. Boston, Ginn, 1913. 6+61 pp. $0.50

MAITLAND (F.). See POINCARÉ (H.).

MANNING (H. P.). Geometry of four dimensions. New York, Macmillan, 1914. 8vo. 10+348 pp.

MARCOLONGO (R.). See BURALI-FORTI (C.).

NIEWENGLOWSKI (B.). Cours de géométrie analytique. 2e édition. Tome III: Géométrie dans l'espace (avec une note de E. Borel, sur les transformations en géométrie.) Paris, Gauthier-Villars, 1914. 8vo. 4+608 pp. Fr. 14.00

PICARD (E.). L'oeuvre de Henri Poincaré. (*Annales scientifiques de l' Ecole Normale Supérieure.*) Paris, Gauthier-Villars, 1913. 4to. 22 pp. Fr. 2.00

POINCARÉ (H.). Wissenschaft und Hypothese. Autorisierte deutsche Ausgabe mit erläuternden Anmerkungen von F. und L. Lindemann. 3. Auflage. (Wissenschaft und Hypothese, Band L.) Leipzig, Teubner, 1914. 8vo. 16+346 pp. M. 4.80

——. Science and method. Translated by Francis Maitland with an introduction by the Hon. Bertrand Russell. London, Nelson, 1914. 6s.

QUESTIONI riguardanti le matematiche elementari. Articoli di Baroni, Daniele, Vitali, ed altri, raccolte e coordinate da F. Enriques. Volume II: Problemi classici della geometria; numeri primi e analisi indeterminata; massimi e minimi. Bologna, Zanichelli, 1914. 8vo. 8+811 pp. L. 30.00

ROYCE (K. and J.). See ENRIQUES (F.).

RUSSELL (B.). See POINCARÉ (H.).

SALOMON (C.). Nouveaux essais de magie arithmétique polygonale. Etoiles magiques à 10 et 12 branches (30, 36, 48 points) et hexagones

et octogones magiques. Paris, Gauthier-Villars, 1913. 8vo. 26 pp. Fr. 1.50

SCHLESINGER (L.). See KLEIN (F.).

SCHLÖMILCH (O.). See FORT (O.).

SMART (E. H.). First course in projective geometry. London, Macmillan, 1913. 24+273 pp. Cloth. 7s. 6d.

SOMMERVILLE (D. M. Y.). The elements of non-euclidean geometry. (Bell's Mathematical Series.) London, Bell, 1914. 8vo. 16+274 pp. Cloth. 5s.

VITALI (G.). See QUESTIONI.

VOLTERRA (V.). Henri Poincaré. L'oeuvre scientifique, l'oeuvre philosophique, par Vito Volterra, Jacques Hadamard, Paul Langevin, Pierre Boutroux. (*Revue du Mois.*) (Nouvelle collection scientifique; directeur, E. Borel.) Paris, Alcan, 1914. 16mo. Fr. 3.50

WANGERIN (A.). See ENCYCLOPÉDIE.

WATSON (G. N.). Complex integration and Cauchy's theorem. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 15.) Cambridge, University Press, 1914. 8vo. 79 pp. 3s.

WIELEITNER (H.). Algebraische Kurven. Neue Bearbeitung. I. Teil: Gestaltliche Verhältnisse. (Sammlung Göschen, Nr. 435.) Leipzig, Göschen, 1914. 12mo. 146 pp. M. 0.90

II. ELEMENTARY MATHEMATICS.

BARNARD (S.) and CHILD (J. M.). Key to "A new algebra," volume 2, containing parts IV, V and VI. London, Macmillan, 1913. Pp. 447–915. Cloth. 8s. 6d.

BECK (H.). See STÄCKEL (P.).

BERNARDI (G.). Soluzionari degli esercizi di trigonometria piana contenuti nel Trattato di trigonometria rettilinea e sferica di J. A. Serret (Edizione Le Monnier). 3a edizione nuovamente migliorata. Firenze, Prezzo, 1913. 16mo. 8+110 pp. L. 1.25

BOREL (E.). See STÄCKEL (P.).

BOS (H.) et REBIÈRE (A.). Eléments de géométrie. 7e édition. Paris, 1913. 8vo. Fr. 7.00

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CARSON (G. ST. L.) and SMITH (D. E.). Elements of algebra. Part I. Boston, Ginn, 1914. 5+346 pp. $0.75

CHILD (J. M.). See BARNARD (S.).

DELENS (P.). Problèmes d'arithmétique amusante. Paris, Vuibert, 1914. 8vo. 8+164 pp. Fr. 2.00

DRESSLER (H.). See INTERNATIONALE.

FITZ-PATRICK (J.). Exercices d'arithmétique, énoncés et solutions. Avec une préface de J. Tannery. 3e édition entièrement refondue et considérablement augmentée. Paris, Hermann, 1914. 8vo. 8+710 pp. Fr. 12.00

FREEMAN (H.). Arithmetic. London, Bell, 1914. 8vo. 2s. 6d.

GELIN (E.). Précis d'arithmétique élémentaire. 6e édition. Namur, Wesmael-Charlier, 1914. 8vo. 288 pp. Fr. 3.00

GOODACRE (H. H.), HOLMES (E. F.), NOBLE (C. F.) and STEER (P.). Bell's outdoor and indoor experimental arithmetic. First year's course (Standard III), 30 pp., price 3d. and 4d. Second year's course (Standard IV), 32 pp., price 3d. and 4d. Third year's course (Standard V), 39 pp., price 3d. and 4d. Fourth year's course (Standard VI), 39 pp., price 4d. and 6d. Fifth year's course (Standard VIII), 48 pp., price 4d. and 6d. London, Bell, 1913.

GUILLEMIN (A.). Tables de logarithmes à 3 quatrades et les nombres correspondants avec 12–13 chiffres. Système normal. Paris, Gauthier-Villars, 1912. 8vo. 22+102+26 pp. Fr. 6.00

HEGER (R.). Fünfstellige logarithmetische und goniometrische Tafeln, sowie Hülfstafeln zur Auflösung höherer numerischer Gleichungen. 2te Auflage. Leipzig, 1914. 8vo. 4+124 pp. Cloth. M. 2.40

HESS (A.). Planimetrie mit einem Abriss über die Kegelschnitte. Berlin, Springer, 1914. 162 pp. M. 2.80

HOLBORN (L.) und SCHEEL (K.). Vier- und fünfstellige Logarithmentafeln, nebst einigen physikalischen Konstanten. 2te Auflage. Braunschweig, 1914. 8vo. 24 pp. Boards. M. 0.80

HOLDER (O.). Die Arithmetik in strenger Begründung. Leipzig, 1914. 4to. 4+74 pp. Boards. M. 3.60

HOLMES (E. F.). See GOODACRE (H. H.).

INTERNATIONALE Mathematische Unterrichtskommission. Berichte und Mitteilungen, veranlasst durch die Internationale Mathematische Unterrichtskommission. VIII. Enthaltend: P. Stäckel, Nachruf auf Peter Treutlein; W. Lietzmann, Der Internationale Mathematikerkongress in Cambridge. Leipzig, Teubner, 1913. 8vo. 57 pp. M. 1.60

——. Berichte und Mitteilungen, veranlasst durch die Internationale Mathematische Unterrichtskommission. IX. Enthaltend: H. Dressler, Mathematische Lehrmittelsammlungen insbesondere für höhere Schulen. Leipzig, Teubner, 1913. 8vo. 30 pp. M. 1.00

JAKOB (J.). Praktische Methodik des mathematischen Unterrichts. Wien, Pichler, 1913. 174 pp. M. 3.40

LIETZMANN (W.). See INTERNATIONALE.

MAGGI (G. A.). Geometria del movimento: lezioni di cinematica, con un' appendice sulla geometria della massa. Pisa, Spoerri, 1914. 8vo. 5+225 pp. L. 8.00

MILNE (R. M.). Mathematical papers for admission into the Royal Military Academy and the Royal Military College for the years 1905–1913. Edited by R. M. Milne. London, Macmillan, 1914. 6s.

MÜLLER (H.). Die Mathematik auf den Gymnasien und Realschulen. Ausgabe B: Für reale Anstalten und Reformschulen. Teil I: Unterstufe. 8. Auflage. Leipzig, 1914. 8vo. 8+208 pp. Cloth. M. 2.20

NOBLE (C. F.). See GOODACRE (H. H.).

REBIÈRE (A.). See BOS (H.).

SCHEEL (K.). See HOLBORN (L.).

SERRET (J. A.). See BERNARDI (G.).

SMITH (D. E.). See CARSON (G. ST. L.).

STÄCKEL (P.). See INTERNATIONALE.

STÄCKEL (P.) und BECK (H.). Lösung der Aufgaben aus Borel-Stäckel Elemente der Mathematik. Heft I: Aufgaben aus der Arithmetik und der Algebra. 8vo. 2+44 pp. Heft II: Aufgaben aus der Geometrie. 8vo. 2+40 pp. Leipzig, Teubner, 1913. M. 1.50+1.50

STEER (P.). See GOODACRE (H. H.).

TANNERY (J.). See FITZ-PATRICK (J.).

VERSLUYS (J.). Zes en negentig Bewijsen van het Theorema van Pythagoras. Amsterdam, A. Versluys, 1914. 8vo. 104 pp. Fl. 1.00

III. APPLIED MATHEMATICS.

APPELL (P.). See ENCYCLOPÉDIE.

D'ARCY (P.). See NEWTON (I.).

ASTRONOMISCHER Kalender für 1914. Herausgegeben von der K. K. Sternwarte. 3. Folge, 4. Jahrgang. Wien, Gerold, 1914. 147 pp. M. 3.00

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